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Building good deals with arbitrage-free discrete time pricing models

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ABSTRACT

Recent literature has proved that many classical very important pricing models of Financial Economics (Black and Scholes, Heston, etc.) and risk measures (*VaR*, *CVaR*, etc.) may lead to "pathological meaningless situations", since there exist sequences of portfolios whose negative risk and positive expected return are unbounded. Such a sequence of strategies will be called "good deal".

This paper focuses on a discrete time arbitrage-free and complete pricing model and goes beyond existence properties. It deals with the effective construction of good deals, i.e., sequences $(y_m)_{m=1}^{\infty}$ of portfolios such that

$(VaR(y_m), CVaR(y_m), Expected\_return(y_m))$

tends to $(-\infty, -\infty, +\infty)$. Under quite general conditions the explicit expression of a good deal is given, and practical algorithms are provided. The sensitivity of our results with respect to measurement errors or dynamic changes of the parameters is analyzed, and numerical experiments are presented with the binomial model.

1. Introduction

Risk measurement is becoming more and more important in financial literature. A clear proof is the growing interest of many authors about the formal properties that a risk measure must satisfy. Indeed, among other interesting contributions, Artzner et al. (1999) introduced the Coherent Measures of Risk, Goovaerts et al. (2004) introduced the Consistent Risk Measures, Rockafellar et al. (2006) defined the Expectation Bounded Risk Measures, Balbás et al. (2009) studied the Adapted Risk Measures, Aumann and Serrano (2008) and Bali et al. (2011) defined Indexes of Riskiness, and Cerreia-Vioglio et al. (2011) defined the Cash-Sub-Additive and Quasi-Convex Risk Measures. All these measures are more and more used by researchers, practitioners, regulators and supervisors.

Many authors have revisited the most important classical actuarial and financial problems by drawing on the risk measures above. With respect to the Portfolio Choice Problem, interesting contributions may be found in Stoyanov et al. (2007), Rockafellar et al. (2007), Miller and Ruszczynski (2008), Zakamouline and Koekebbaker (2009), and Balbás et al. (2010a), among others. Usually, authors attempt to maximize a generalized Sharpe ratio or deal with a vector optimization problem involving the expected return and a (maybe vector) risk measure. In this sense, they extend the Markowitz approach, but the role of the standard deviation is played by a more complex risk measure.

A second recent line of research focuses on the notion of "Good Deal" (*GD*), introduced in Cochrane and Saa-Requejo (2000). Mainly, a *GD* is an investment strategy providing traders with a "very high Sharpe ratio", in comparison with the Market Portfolio. In the paper of Cochrane and Saa-Requejo the risk was measured with the standard deviation, and the absence of *GD* was imposed in an arbitrage-free incomplete pricing model so as to price non-reachable pay-offs. Unreachable pay-offs are priced in such a manner that buyers (sellers) cannot create a *GD* with huge Sharpe ratios provoked by low (high) bid (ask) prices. In practice, the absence of *GD* generates bid/ask spreads (or good deal bounds) much lower than those implied by arbitrage arguments. Consequently, many researchers have extended the discussion and dealt with *GD*-linked pricing methods, obtaining *GD* bounds and hedging strategies much more realistic and empirically relevant than the classical arbitrage-linked bounds and hedging strategies. Moreover, this line of research has been generalized for risk measures beyond the standard deviation (among others, Staum, 2004, or Arai, 2011, present further studies about all of these topics).

When portfolio choice problems do not focus on the standard deviation and minimize other risk measures, it is not guaranteed that the problem will be bounded. In particular, Balbás et al. (2010a) have shown that every pricing model whose Stochastic Discount Factor (*SDF*) follows a log-normal or a heavier-tailed distribution (Black and Scholes, Heston, etc.) will generate meaningless situations when combined with every coherent and

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expectation bounded risk measure. Indeed, for every pricing model and risk measure as above, there are sequences of portfolios whose risk tends to minus infinity or remains bounded and whose expected return tends to plus infinity (*risk* = $-\infty$, *return* = $+\infty$ or *risk* $\leq$ *CONSTANT*, *return* = $+\infty$). The analysis of Balbás et al. (2010a) has been extended in Balbás et al. (2010b), where the authors present explicit constructions of the sequences above for the Conditional Value at Risk (*CVaR*) with an arbitrary level of confidence and the Black and Scholes model. Balbás et al. (2010b) use the expression *GD* to indicate such a sequence. Actually, according to the results above, the generalized Sharpe ratio "*Return*/*CVaR*" may tend to infinity, and therefore it will outperform the Market Portfolio generalized Sharpe ratio. Since the ratio may become as close as desired to infinity, the notion of *GD* in Balbás et al. (2010b) is obviously more restrictive than it is in the papers above (Cochrane and Saa-Requejo, 2000; Staum, 2004; Arai, 2011). Surprisingly, despite the fact that the absence of *GD* may be useful to price in incomplete models, complete ones such Black and Scholes are not *GD*-free for general risk measures beyond the standard deviation. This paradox implies that the explicit computation of *GD* in complete models becomes a critical point, since it may help to study what is the economic/financial meaning of the *GD*-absence assumption.

This paper adopts the notion of *GD* of Balbás et al. (2010b) and deals with the Value at Risk (*VaR*) and the *CVaR* so as to present effective *GD* constructions for every discrete time arbitrage-free and complete pricing model such that the existence of *GD* holds. In other words, for an arbitrary (discrete) pricing model, if there are sequences $(y_m)_{m=1}^{\infty}$ of pay-offs such that

$$(VaR(y_m), CVaR(y_m), Expected\_return(y_m)) \longrightarrow (-\infty, -\infty, +\infty)$$

holds, then they can be computed by the algorithms that we will present. Moreover, as a second contribution, we will provide a sensitivity analysis, i.e., we will measure the effect on the *GD* of estimation errors or dynamic evolutions of some key variables such as the *SDF* of the pricing model.

We have selected *VaR* and *CVaR* because they are becoming more and more popular for researchers, practitioners, regulators and supervisors, and they are also playing an important role in international regulations such as Basel II and III.[1] We have selected a discrete time framework because it significantly simplifies the mathematical exposition of the paper. Moreover, since most of the continuous time pricing models have an appropriate discrete time approximation, it seems that the provided algorithms may be quite useful to traders in practice. Needless to say that return/risk ratios are crucial in order to rank the effectiveness of portfolio managers.

The article's outline is as follows. Section 2 will present our notations and a important background that will be applied. We will deal with the framework of Balbás et al. (2010b), so some further details may be found in that paper. There are no contributions in this section, but it has been included for expositional simplicity. Section 3 is the most important one of the paper. In particular, Theorems 4 and 5 will yield closed formulas providing us with a *GD* in a very general discrete time setting. They are crucial to develop two new algorithms in Remarks 3 and 4. Four numerical experiments with the popular binomial pricing model will be summarized in Section 4. They mainly have illustrative purposes, though the fourth one involves real market data related to the American index *SP*500 during the year 2011. We did not implement any exhaustive empirical test, but, for the index and year above, the *GD* practical performance was quite satisfactory. Besides, the third numerical example will show that the algorithms are flexible enough, in the sense that they may dynamically incorporate market evolutions that the pricing model does not predict (modifications in volatilities, interest rates, etc.). Nevertheless, since it may be also interesting to anticipate these changes and have initial information about their possible effect on the *GD*, Section 5 is devoted to measuring the sensitivity of our solutions with respect to them, as well as the sensitivity with respect to possible measurement errors. Theorem 6 gives a general formula when the *GD* price, the random final wealth of the manager, and/or the pricing model (the *SDF*) are modified. Finally, Section 6 presents the most important conclusions of the paper.

2. Preliminaries, notations and theoretical background

Consider the probability space $(\Omega, \mathcal{F}, \mathrm{IP})$ composed of a finite set of "states of the world" Ω, the σ-algebra $\mathcal{F}$ containing all the subsets of Ω, and the probability measure IP whose support is Ω. Consider also a time interval $[0, T]$, a finite subset $\mathcal{T} \subset [0, T]$ of trading dates containing 0 and T, and a filtration $(\mathcal{F}_t)_{t\in\mathcal{T}}$ providing the arrival of information such that $\mathcal{F}_0 = \{\emptyset, \Omega\}$ and $\mathcal{F}_T = \mathcal{F}$. As usual, there are several available securities whose price processes are adapted to $(\mathcal{F}_t)_{t\in\mathcal{T}}$. Suppose that the market is complete, i.e., every final pay-off (or $\mathcal{F}$-measurable random variable) $y \in \mathrm{IR}^{\Omega}$ may be reached by the price process $(S_t)_{t\in\mathcal{T}}$ of a self-financing portfolio, in the sense that the equality $S_T = y$ holds. Then, S_0 may be interpreted as the initial (at $t = 0$) price $\Pi(y)$ of the pay-off y, and we will assume that the pricing rule $\Pi : \mathrm{IR}^{\Omega} \longrightarrow \mathrm{IR}$ is linear (i.e., there are no frictions).

The completeness of the pricing model implies the existence of a risk-free asset. Thus, if $r_f \geq 0$ is the risk-free rate, equality

$$\Pi(k) = ke^{-r_f T} \tag{1}$$

must hold for every $k \in \mathrm{IR}$. Besides, according to the Riesz Representation Theorem, there exists a unique $z_\pi \in \mathrm{IR}^{\Omega}$ such that

$$\Pi(y) = e^{-r_f T}\mathrm{IE}(yz_\pi) \tag{2}$$

for every $y \in \mathrm{IR}^{\Omega}$, IE() representing the mathematical expectation. Moreover, to prevent the existence of arbitrage, the strict inequality

$$z_\pi > 0 \tag{3}$$

must hold. z_π is usually called "Stochastic Discount Factor" (*SDF*, see Duffie, 1988, for further details).

Expressions (1) and (2) imply that $ke^{-r_f T} = \Pi(k) = e^{-r_f T}k\mathrm{IE}(z_\pi)$, which leads to

$$\mathrm{IE}(z_\pi) = 1. \tag{4}$$

We will deal with two risk measures in this paper: The Value at Risk (VaR_{μ_0}) and the Conditional Value at Risk ($CVaR_{\mu_0}$), with $\mu_0 \in (0, 1)$ being the confidence level. They are given by (Rockafellar et al., 2006)

$$VaR_{\mu_0}(y) = -Inf\,\{V \in \mathrm{IR};\ \mathrm{IP}(y \leq V) > 1 - \mu_0\}$$

and

$$CVaR_{\mu_0}(y) = \frac{1}{1-\mu_0}\int_0^{1-\mu_0} VaR_{1-t}(y)dt$$

for every $y \in \mathrm{IR}^{\Omega}$. Moreover, if the $CVaR_{\mu_0}$ sub-gradient is defined by

$$\Delta_{CVaR_{\mu_0}} = \left\{z \in \mathrm{IR}^{\Omega};\ \mathrm{IE}(z) = 1,\ 0 \leq z \leq \frac{1}{1-\mu_0}\right\}. \tag{5}$$

then

$$CVaR_{\mu_0}(y) = Max\left\{-\mathrm{IE}(yz);\ z \in \Delta_{CVaR_{\mu_0}}\right\} \tag{6}$$

[1] Moreover, Ogryczak and Ruszczynski (2002) have shown that *CVaR* is consistent with the second order stochastic dominance and the usual utility functions.

holds for every $y \in \mathrm{IR}^{\Omega}$ (Rockafellar et al., 2006). Finally, if $\rho = VaR_{\mu_0}$ or $\rho = CVaR_{\mu_0}$ for some level of confidence $0<\mu_0<1$, all of the properties below hold:

$$CVaR_{\mu_0}(y) \geq VaR_{\mu_0}(y) \tag{7}$$

for every $y \in \mathrm{IR}^{\Omega}$,

$$\rho(y+k) = \rho(y) - k \tag{8}$$

for every $y \in \mathrm{IR}^{\Omega}$ and $k \in \mathrm{IR}$,

$$\rho(\alpha y) = \alpha \rho(y) \tag{9}$$

for every $y \in \mathrm{IR}^{\Omega}$ and $\alpha > 0$,

$$CVaR_{\mu_0}(y_1+y_2) \leq CVaR_{\mu_0}(y_1) + CVaR_{\mu_0}(y_2) \tag{10}$$

for every $y_1, y_2 \in \mathrm{IR}^{\Omega}$,

$$CVaR_{\mu_0}(y) \geq -\mathrm{IE}(y) \tag{11}$$

for every $y \in \mathrm{IR}^{\Omega}$, and

$$\rho(y_1) \geq \rho(y_2) \tag{12}$$

for every $y_1, y_2 \in \mathrm{IR}^{\Omega}$ with $y_1 \leq y_2$.[2]

Suppose that the random variable $y_0 \in \mathrm{IR}^{\Omega}$ represents a trader's final (at T) wealth (or pay-off). The risk level is given by $\rho(y_0)$. Then $\rho(y_0)$ may be an adequate final value (at T) of the capital requirement. Indeed, (8) leads to

$$\rho(y_0 + \rho(y_0)) = 0 \tag{13}$$

and the risk will vanish if the additional amount $\rho(y_0)e^{-r_f T}$ is invested in the risk-free security. Nevertheless, Balbás et al. (2010b) have proved that this investment in the risk-free security may be outperformed by alternative hedging strategies $y \in \mathrm{IR}^{\Omega}$, in the sense that the current price of y is still $\rho(y_0)e^{-r_f T}$ but the global risk $\rho(y_0+y)$ is negative. More accurately, these authors consider the pay-off $y \in \mathrm{IR}^{\Omega}$ added by the trader to his pay-off $y_0 \in \mathrm{IR}^{\Omega}$, they suppose that

$$C > 0 \tag{14}$$

gives (the value at T of) the highest amount of money devoted to reducing the risk level,[3] and they finally propose the following optimization problems so as to select y:

$$\begin{cases} Min\, CVaR_{\mu_0}(y + y_0 - \mathrm{IE}(yz_\pi)) \\ \mathrm{IE}(yz_\pi) \leq C \\ y \geq 0 \end{cases} \tag{15}$$

and

$$\begin{cases} Min\, CVaR_{\mu_0}(y + y_0 - \mathrm{IE}(yz_\pi)) \\ \mathrm{IE}(yz_\pi) \leq C \end{cases} \tag{16}$$

Problem (15) involves the global risk $CVaR_{\mu_0}(y + y_0 - \mathrm{IE}(yz_\pi))$ that the trader is facing, so it has to incorporate the value $\mathrm{IE}(yz_\pi)$ of the added portfolio, that will have to be paid and will reduce the trader's wealth. Constraint $y \geq 0$ may be indicating the presence of short-selling restrictions. Since we are minimizing risk, one can consider that short sales must be allowed if they do not make the riskiness increase, so Problem (16) also makes sense. In fact, the optimal risk in (16) will never be higher than the optimal risk in (15), since every (15)-feasible solution is also (16)-feasible.

Since $y=0$ satisfies the constraints of (15) and (16) both problems are feasible. However, the paper above presents examples illustrating that (16) may be unbounded, i.e., there may be sequences $(y_n)_{n=1}^{\infty}$ of pay-offs such that $CVaR_{\mu_0}(y_n + y_0 - \mathrm{IE}(yz_\pi)) \to -\infty$. Furthermore, as we will prove in Proposition 1 below, if the existence of this sequence holds then it provides us with returns converging to $+\infty$. Henceforth, these sequences will be called good deals (*GD*).

Proposition 1. *If the sequence* $(y_n)_{n=1}^{\infty}$ *satisfies* $Lim_{n\to\infty} CVaR_{\mu_0}(y_n + y_0 - \mathrm{IE}(y_n z_\pi)) = -\infty$, *then*

$$Lim_{n\to\infty} VaR_{\mu_0}(y_n + y_0 - \mathrm{IE}(y_n z_\pi)) = -\infty$$

and

$$Lim_{n\to\infty} \mathrm{IE}(y_n + y_0 - \mathrm{IE}(y_n z_\pi)) = +\infty.$$

Proof. (11) shows that

$$\mathrm{IE}(y_n + y_0 - \mathrm{IE}(y_n z_\pi)) \geq -CVaR_{\mu_0}(y_n + y_0 - \mathrm{IE}(y_n z_\pi)) \to +\infty.$$

Besides, (7) shows that

$$VaR_{\mu_0}(y_n + y_0 - \mathrm{IE}(y_n z_\pi)) \leq CVaR_{\mu_0}(y_n + y_0 - \mathrm{IE}(y_n z_\pi)) \to -\infty.$$

□

Following Balbás et al. (2010b), the solution y^* of (15), if it exists, will be called "shadow riskless asset" (*SRA*).

The rest of this section is devoted to summarizing some findings of Balbás et al. (2010b) that will apply henceforth. In particular, (6) implies that

$$\begin{cases} Max\ -C\lambda - \mathrm{IE}(y_0 z) \\ z \leq (1+\lambda) z_\pi \\ \lambda \in \mathrm{IR},\ \lambda \geq 0,\ z \in \Delta_{CVaR_{\mu_0}} \end{cases} \tag{17}$$

is the dual of (15), $\lambda \in \mathrm{IR}$ and $z \in \Delta_{CVaR_{\mu_0}}$ being the decision variables.

Theorem 2. *Suppose that* $\mathrm{IP}\left(z_\pi > \frac{1}{1-\mu_0}\right) > 0$.[4] *Then:*

(a) *Problem* (16) *is unbounded, i.e., there are good deals.*
(b) (15) *and* (17) *are bounded and solvable, and there is no duality gap (i.e., both problems attain their common optimal value). If* $y^* \in \mathrm{IR}^{\Omega}$ *and* $(\lambda^*, z^*) \in \mathrm{IR} \times \mathrm{IR}^{\Omega}$, *then they solve* (15) *and* (17) *if and only if there exist* $\alpha \in \mathrm{IR}$, $\alpha_1, \alpha_2 \in \mathrm{IR}^{\Omega}$ *and a disjoint partition* $\Omega = \Omega_0 \cup \Omega_1 \cup \Omega_2$ *such that the following Karush–Kuhn–Tucker conditions*

$$\begin{cases} C - \mathrm{IE}(y^* z_\pi) = 0 \\ y^* + y_0 = \alpha - \alpha_1 + \alpha_2 \\ \alpha_i \geq 0, \quad i = 1, 2 \\ \alpha_1 = \alpha_2 = 0 \quad \text{on } \Omega_0 \\ z^* = \dfrac{1}{1-\mu_0} \text{ and } \alpha_2 = 0 \quad \text{on } \Omega_1 \\ z^* = 0 \textit{ and } \alpha_1 = 0 \quad \text{on } \Omega_2 \\ ((1+\lambda^*) z_\pi - z^*) y^* = 0 \\ (1+\lambda^*) z_\pi - z^* \geq 0 \\ y^* \in \mathrm{IR}^{\Omega},\ y^* \geq 0,\ \lambda^* \in \mathrm{IR},\ \lambda^* \geq 0,\ z^* \in \Delta_{CVaR_{\mu_0}} \end{cases} \tag{18}$$

[2] According to Artzner et al. (1999) and Rockafellar et al. (2006), a risk measure satisfying (8), (9)–(12) is said to be coherent and expectation bounded. $CVaR_{\mu_0}$ is a very important example.

[3] If $\rho(y_0) > 0$ then (13) shows that $C = \rho(y_0)$ could be a suitable choice for C.

[4] Bearing in mind (3)–(5), $\mathrm{IP}(z_\pi > 1/(1-\mu_0)) > 0$ if and only if $z_\pi \notin \Delta_{CVaR_{\mu_0}}$, i.e., the *SDF* is not in the $CVaR_{\mu_0}$ sub-gradient.

hold. Moreover, the solution y^* of (15) is not a risk-free asset and the solution (λ^*, z^*) of (17) satisfies $\lambda^* > 0$

3. Constructing good deals

Conditions (18) may be complex in practical applications. Thus, let us simplify them. We will look for the optimal solution of (15) among the non-path-dependent European style derivatives with maturity at T and whose underlying asset is y_0. The obvious implication is that the final pay-off of every non-path-dependent European style derivative is a function of y_0, so $\omega \in \Omega$ is not so important and only $y_0(\omega)$ matters. The set of states on nature Ω may be replaced by a new set $\Omega^* \subset \Omega$ only distinguishing the final values of y_0, and Π may be replaced by its restriction too. Illustrative examples will be given in Section 4. Proposition 1 and Theorem 2 still applies if Ω^* plays the role of Ω. Clearly, once the solution y^* of (15) has been obtained, the self-financing trading strategy leading to the pay-off y^* will have to be constructed in the initial setting $(\Omega, \mathcal{F}, \mathrm{IP})$. Once again, the examples of Section 4 will clarify it.

Hence, without loss of generality we can consider that

$$\Omega^* = \left\{0, 1, 2, \ldots, n\right\}, \tag{19}$$

and $y_0(\omega)$ increases as so does $\omega \in \Omega^*$. We will also assume that

$$z_\pi(0) > \frac{1}{1-\mu_0} \tag{20}$$

and $z_\pi(\omega)$ decreases as so does $\omega \in \Omega^*$. Once again, Section 4 will clarify that z_π usually decreases in practical applications such that y_0 increases.

Summarizing we have:

Assumption 1. Ω^* is given by (19), y_0 is a strictly increasing function of $\omega \in \Omega^*$, z_π is a strictly decreasing function of $\omega \in \Omega^*$, and (20) holds.[5] □

Remark 1. Under the conditions above, Theorem 2a implies the existence of good deals, while Theorem 2b implies the existence of both a dual solution (λ^*, z^*) and a *SRA* y^* which is not risk-free. Furthermore, Theorem 2b also leads to both the equality

$$C - \mathrm{IE}(y^* z_\pi) = 0 \tag{21}$$

and the inequality

$$\lambda^* > 0. \tag{22}$$

□

Let us give several properties that will allow us to solve (15) and (16).[6] First of all, though (15) and (16) are not linear, Problem (17) is linear, and it can be solved by standard well-known methods. Thus we can assume that (λ^*, z^*) is known.

Theorem 3. *z^* is decreasing, i.e., if $\omega, \tilde{\omega} \in \Omega^*, \omega < \tilde{\omega}$, then $z^*(\omega) \geq z^*(\tilde{\omega})$.*

Proof. Suppose that $z^*(\omega) < z^*(\tilde{\omega})$. Then, since z_π is strictly decreasing, the ninth condition in (18) leads to

$$(1+\lambda^*) z_\pi(\omega) > (1+\lambda^*) z_\pi(\tilde{\omega}) \geq z^*(\tilde{\omega}) > z^*(\omega),$$

and the eighth condition in (18) implies that $y^*(\omega) = 0$. Besides, $\omega \notin \Omega_1$ in (18) and $\tilde{\omega} \notin \Omega_2$. Hence,

$$\begin{aligned} y_0(\tilde{\omega}) \leq y_0(\tilde{\omega}) + y^*(\tilde{\omega}) &= \alpha - \alpha_1(\tilde{\omega}) \leq \alpha \leq \alpha + \alpha_2(\omega) \\ &= y_0(\omega) + y^*(\omega) = y_0(\omega), \end{aligned}$$

which contradicts that y_0 is strictly increasing. □

Remark 2. Expression (20) implies the existence of $\omega \in \Omega^*$ with $z_\pi(\omega) > \frac{1}{1-\mu_0}$. Since $\lambda^* > 0$ (see (22)), we have that

$$(1+\lambda^*) z_\pi(\omega) > \frac{1}{1-\mu_0}$$

must hold for some $\omega \in \Omega^*$. Henceforth we will fix

$$\omega_0 = Max\left\{\omega \in \Omega^*;\ (1+\lambda^*) z_\pi(\omega) > \frac{1}{1-\mu_0}\right\}. \tag{23}$$

Similarly, if $\mathrm{IP}(z^* > 0) < 1$, there exists

$$\omega_1 = Min\left\{\omega \in \Omega;\ z^*(\omega) = 0\right\}, \tag{24}$$

and we will define $\omega_1 = n+1$ if $\mathrm{IP}(z^* > 0) = 1$. □

Next let us show that $\left\{0, 1, \ldots, \omega_0\right\}$ and $\{\omega_1, \ldots, n\}$ are disjoint, along with the expression of y^* and z^* in these subsets of Ω^*.

Theorem 4.

(a) *If $\mathrm{IP}(z^* > 0) < 1$ then*

$$\begin{cases} z^* = 0, & \omega \geq \omega_1 \\ z^* > 0, & \omega < \omega_1 \end{cases} \tag{25}$$

(b) *$y^*(\omega) = 0$ for every $\omega \leq \omega_0$.*
(c) *If $\mathrm{IP}(z^* > 0) < 1$ then $y^*(\omega) = 0$ for every $\omega \geq \omega_1$.*
(d) *$\omega_0 < n$, and $\omega_0 + 1 < \omega_1$ if $\mathrm{IP}(z^* > 0) < 1$.*
(e) *$z^*(\omega) = \frac{1}{1-\mu_0}$ for every $\omega \leq \omega_0$.*

Proof.

(a) It trivially follows from Theorem 3 and (24).
(b) Since z_π is decreasing we have that

$$(1+\lambda^*) z_\pi(\omega) > \frac{1}{1-\mu_0} \geq z^*(\omega)$$

for $\omega \leq \omega_0$, and the eighth and ninth conditions in (18) imply that $y^*(\omega) = 0$.
(c) Statement (a) implies that $z^*(\omega) = 0$. Expression (3), along with the eighth and ninth conditions in (18), show that $y^*(\omega) = 0$. Similar arguments show that $\omega_0 < n$.
(d) $\omega_0 + 1 \geq \omega_1$ and Statements (b) and (c) would lead to $y^* = 0$, in contradiction with (21) and (14).
(e) Theorems 3 and (5) imply that one only has to prove $z^*(\omega_0) = \frac{1}{1-\mu_0}$. If $z^*(\omega_0) < \frac{1}{1-\mu_0}$ then $\omega_0 \in \Omega_0$ in (18) due to Statement (d). Hence, according to Statement (b),

$$y_0(\omega_0) = y_0(\omega_0) + y^*(\omega_0) = \alpha.$$

On the other hand, $\omega_0 + 1 \notin \Omega_2$ in (18) due to Statement (d). Hence,

$$y_0(\omega_0+1) \leq y_0(\omega_0+1) + y^*(\omega_0+1) = \alpha - \alpha_1 \leq \alpha.$$

We have a contradiction because y_0 is strictly increasing.

□

[5] Actually, a quite parallel analysis could be implemented if z_π were a strictly increasing function of $\omega \in \Omega^*$, though we do not address this case because it would significantly enlarge the paper. We have chosen a decreasing z_π because it is the usual situation in a risk adverse world, as will be seen in Section 4.

[6] Theorem 2a shows that (16) is unbounded and cannot be solved. However, we will give a concrete sequence of portfolios whose $(VaR_{\mu_0}, CVaR_{\mu_0}, Expected\_return)$ tends to $(-\infty, -\infty, +\infty)$.

The *SRA* y^* is already known in $\{0, 1, \ldots, \omega_0\}$ and $\{\omega_1, \ldots, n\}$, so let us compute y^* within the interval $\omega_0 < \omega < \omega_1$.

Theorem 5.

(a) $$(1+\lambda^*)z_\pi(\omega_0+1) \leq \frac{1}{1-\mu_0}. \tag{26}$$

(b) *If (26) is a strict inequality then*

$$y^* = \begin{cases} 0, & \omega \leq \omega_0 \\ \dfrac{C+\sum_{\omega=\omega_0+1}^{\omega_1-1} y_0(\omega)z_\pi(\omega)\mathrm{IP}(\omega)}{\sum_{\omega=\omega_0+1}^{\omega_1-1} z_\pi(\omega)\mathrm{IP}(\omega)} - y_0, & \omega_0 < \omega < \omega_1 \\ 0, & \omega \geq \omega_1 \end{cases}. \tag{27}$$

(c) *If (26) becomes a equality then there exist $\alpha \geq \tilde{\alpha} > 0$ such that*

$$y^* = \begin{cases} 0, & \omega \leq \omega_0 \\ \tilde{\alpha} - y_0, & \omega = \omega_0 + 1 \\ \alpha - y_0, & \omega_0 + 1 < \omega < \omega_1 \\ 0, & \omega \geq \omega_1 \end{cases}. \tag{28}$$

Moreover,

$$\tilde{\alpha} z_\pi(\omega_0+1)\mathrm{IP}(\omega_0+1) + \alpha\left(\sum_{\omega_0+2<\omega\leq\omega_1-1} z_\pi(\omega)\mathrm{IP}(\omega)\right) = C + \sum_{\omega_0+1\leq\omega\leq\omega_1-1} y_0(\omega)z_\pi(\omega)\mathrm{IP}(\omega). \tag{29}$$

and $\tilde{\alpha} = y_0(\omega_0+1)$ if $z^*(\omega_0+1) < \frac{1}{1-\mu_0}$. Finally, if $z^*(\omega_0+1) = \frac{1}{1-\mu_0}$ then $0 < \tilde{\alpha} \leq \alpha$ may be arbitrary as far as $y^* \geq 0$ and the eighth condition in (18) and (29) hold.

Proof.

(a) It trivially follows from (23).

(b) If (26) is a strict inequality then

$$(1+\lambda^*)z_\pi(\omega) < \frac{1}{1-\mu_0} \tag{30}$$

whenever $\omega_0+1 \leq \omega \leq \omega_1 - 1$ because z_π is decreasing. Hence, the ninth expression in (18) implies that

$$z^*(\omega) < \frac{1}{1-\mu_0} \tag{31}$$

whenever $\omega_0+1 \leq \omega \leq \omega_1 - 1$. Eq. (25) implies that $z^*(\omega) > 0$ for $\omega_0+1 \leq \omega \leq \omega_1 - 1$. Then, "the interval" $\{\omega_0+1, \ldots, \omega_1-1\}$ is included in the set Ω_0 of (18), which implies the existence of $\alpha \in \mathrm{IR}$ such that $y^* = \alpha - y_0$ whenever $\omega_0+1 \leq \omega \leq \omega_1 - 1$. Since Theorem 4 implies that $y^* = 0$ out of this interval, (21) leads to

$$\alpha \sum_{\omega=\omega_0+1}^{\omega_1-1} z_\pi(\omega)\mathrm{IP}(\omega) - \sum_{\omega=\omega_0+1}^{\omega_1-1} y_0(\omega)z_\pi(\omega)\mathrm{IP}(\omega) = C,$$

and (27) becomes obvious.

(c) If (26) becomes a equality then (30) and (31) still hold for $\omega_0+1 \leq \omega \leq \omega_1 - 1$ and $\omega_0+1 < \omega$. As in (b), $y^* = \alpha - y_0$ for $\omega_0+1 \leq \omega \leq \omega_1 - 1$ and $\omega_0+1 < \omega$. As in (b), $z^*(\omega) > 0$ for $\omega_0+1 \leq \omega \leq \omega_1 - 1$, so ω_0+1 does not belong to the set Ω_2 of (18) and $\alpha_2(\omega_0+1) = 0$. Whence

$$y^*(\omega_0+1) = \alpha - \alpha_1(\omega_0+1) - y_0(\omega_0+1).$$

Eq. (28) trivially follows if one takes $\tilde{\alpha} = \alpha - \alpha_1(\omega_0+1)$, which is strictly positive because otherwise $y^*(\omega_0+1)$ would be strictly negative, in contradiction with the constraints of (15). Furthermore, (29) trivially follows from (21), $\tilde{\alpha} = y_0(\omega_0+1)$ if $z^*(\omega_0+1) < 1/(1-\mu_0)$ due to the eighth condition in (18), and finally, we only must guarantee the fulfillment of (18) if $z^*(\omega_0+1) = 1/(1-\mu_0)$.

□

Remark 3. (General algorithm to build the *SRA*). Theorem 5 above allows us to find y^* in practice. The steps are:

Step a Solve the linear programming problem (17) by standard methods (for instance, the simplex method). Theorem 2b guarantees that it is bounded and solvable. We have the dual solution (λ^*, z^*).

Step b Compute ω_0 and ω_1 according to (23) and (24). Take $\omega_1 = n+1$ if $\mathrm{IP}(z^* > 0) = 1$.

Step c Take $y^*(\omega) = 0$ for $\omega \leq \omega_0$. If $\mathrm{IP}(z^* > 0) < 1$ then take $y^*(\omega) = 0$ for every $\omega \geq \omega_1$ (Theorem 4).

Step d Verify whether (26) is a equality or a strict inequality.

Step e If (26) is a strict inequality then take (Theorem 5)

$$y^* = \begin{cases} 0, & \omega \leq \omega_0 \\ k - y_0, & \omega_0 < \omega < \omega_1 \\ 0, & \omega \geq \omega_1 \end{cases}, \tag{32}$$

where

$$k = \frac{C+\sum_{\omega=\omega_0+1}^{\omega_1-1} y_0(\omega)z_\pi(\omega)\mathrm{IP}(\omega)}{\sum_{\omega=\omega_0+1}^{\omega_1-1} z_\pi(\omega)\mathrm{IP}(\omega)}.$$

Step f If (26) were a equality then take y^* as in (28) and (29) (Theorem 5).

□

Remark 4. (General algorithm to build a *GD*). Let us assume that there are no short selling restrictions, i.e., let us deal with (16) rather than (15). Theorem 2a shows that there is a *GD*. In other words, one can construct sequences of portfolios whose

$(VaR_{\mu_0}, CVaR_{\mu_0}, return)$

tends to $(-\infty, -\infty, +\infty)$. Hence, let us give an effective construction of such a sequence.

Consider $m \in \mathrm{IN}$, along with an approximation of (16) given by Problem

$$\begin{cases} Min\ CVaR_{\mu_0}(y + y_0 - \mathrm{IE}(yz_\pi)) \\ \mathrm{IE}(yz_\pi) \leq C \\ y \geq -m \end{cases} \tag{33}$$

Then, due to (4), it is easy to see that the change of variable $x_m = y + m$ leads to

$$\begin{cases} Min\ CVaR_{\mu_0}(x_m + y_0 - \mathrm{IE}(x_m z_\pi)) \\ \mathrm{IE}(x_m z_\pi) \leq C + m \\ x_m \geq 0 \end{cases}, \tag{34}$$

which is a new problem analogous to (15). Thus, (33) is bounded and achieves its optimal value (Theorem 2b). Consider the sequence $(y_m^*)_{m=1}^\infty = (x_m^* - m)_{m=1}^\infty$ of solutions of (33), $(x_m^*)_{m=1}^\infty$ denoting the solutions of (34). It is easy to see that $(y_m^*)_{m=1}^\infty$ is a *GD*. Furthermore, every x_m^* may be computed with the algorithm of Remark 3. Notice that (4) and (8) lead to

$$VaR_{\mu_0}(y_m^* + y_0 - \mathrm{IE}(y_m^* z_\pi)) = VaR_{\mu_0}(x_m^* + y_0 - \mathrm{IE}(x_m^* z_\pi))$$

and

$$CVaR_{\mu_0}(y_m^* + y_0 - \mathrm{IE}(y_m^* z_\pi)) = CVaR_{\mu_0}(x_m^* + y_0 - \mathrm{IE}(x_m^* z_\pi)).$$

Since the limit equality

$$(VaR_{\mu_0}(y_m^* + y_0 - \mathrm{IE}(y_m^* z_\pi)), CVaR_{\mu_0}(y_m^* + y_0 - \mathrm{IE}(y_m^* z_\pi)),$$
$$return(y_m^* + y_0 - \mathrm{IE}(y_m^* z_\pi))) = (-\infty, -\infty, +\infty)$$

is obviously unreachable, in practice one can proceed as follows:

Step a Fix a desired finite level (A, B, C) for $(VaR_{\mu_0}, CVaR_{\mu_0}, return)$.
Step b Apply the algorithm of Remark 3 in order to compute y_m^* for several values of $m \in \mathrm{IN}$, and then stop once $VaR_{\mu_0} \leq A$, $CVaR_{\mu_0} \leq B$, and $return \geq C$.

□

4. Examples and numerical experiments

Assumption 1 usually holds in practice. For instance, suppose that the random pay-off y_0 satisfies the binomial probability distribution, i.e., the price process with final (at T) pay-off y_0 is given by the binomial model with n periods. The set Ω^* of (19) will indicate the number of growths of this price process between 0 and T, and therefore

$$y_0 = Wu^{\omega} d^{n-\omega}, \tag{35}$$

for every $\omega = 0, 1, 2, \ldots, n$, $W > 0$ denoting the price of the portfolio at $t = 0$, and $u > 1$ and $d < 1$ denoting the usual factors affecting this portfolio price between two consecutive trading dates. y_0 is obviously a increasing function of ω. We can also assume that $z_\pi(\omega)$ decreases as $\omega \in \Omega$ increases since this is the usual situation if the market is risk adverse. Indeed, if $\nabla(t)$ represents the time length between consecutive trading dates, and $R = e^{r_f \nabla(t)} \in (d, u)$ represents the capitalization factor of the risk-free asset, then

$$z_\pi = \left(\frac{R-d}{R_{y_0} - d}\right)^{\omega} \left(\frac{u-R}{u - R_{y_0}}\right)^{n-\omega}, \tag{36}$$

R_{y_0} denoting the expected return of y_0 between two consecutive trading dates. Obviously, since y_0 is risky, in an arbitrage-free risk adverse world we have that

$$u > R_{y_0} > R > d > 0,$$

and z_π is decreasing.

Beyond the binomial model, Assumption 1 is usually fulfilled too. Indeed, in a general risk adverse framework, if we assume that $\omega = 0, 1, 2, \ldots, n$ is the number of growths of the price process leading to the pay-off y_0, and this pay-off is efficient in a (*return*, *standard_deviation*) setting, then there exists a couple of strictly positive real numbers η_1 and η_2 such that

$$z_\pi = \eta_1 - \eta_2 y_0, \tag{37}$$

and therefore z_π is strictly decreasing if y_0 is strictly increasing. In practice, η_1 and η_2 may be easily computed from (4) and taking into account that W is the current price of the self-financing portfolio with pay-off y_0. Thus, System

$$\begin{cases} \eta_1 - \eta_2 \mathrm{IE}(y_0) = 1 \\ \eta_1 \mathrm{IE}(y_0) - \eta_2 \mathrm{IE}(y_0^2) = W \end{cases} \tag{38}$$

must hold.

Dealing again with the binomial model, let us present four examples of the two given algorithms (Remarks 3 and 4). Examples 1–3 are just presented for illustrative purposes, whereas Example 4 deals with real market data.[7]

Example 1. Consider that the manager's portfolio price process is given by the binomial model composed of three periods (four trading dates denoted by 0, $\nabla(t)$, $2\nabla(t)$, and $3\nabla(t)$, for some $\nabla(t)$) such that $R = 1.01$, $d = 0.5$ and $u = 2$. Suppose that the initial (at $t = 0$) portfolio price is $W = 200$. The binomial tree below indicates the price process we must deal with

$$\left(\begin{array}{cccc}
\multicolumn{4}{l}{The_price_process_for_the_underlying_asset} \\
 & & & 1600 \\
 & & \nearrow & \\
 & & 800 & \\
 & & \searrow & \\
 & \nearrow & & \\
 & 400 & & 400 \\
 & \searrow & & \\
\nearrow & & \nearrow & \\
200 & & 200 & \\
\searrow & & \searrow & \\
 & \nearrow & & \\
 & 100 & & 100 \\
 & \searrow & & \\
 & & \nearrow & \\
 & & 50 & \\
 & & \searrow & \\
 & & & 25
\end{array}\right) \tag{39}$$

There are $2^3 = 8$ possible trajectories of this price process, so, according to Section 2,

$$\Omega = \{(d, d, d), (d, d, u), (d, u, d), (u, d, d), (d, u, u), (u, d, u),$$
$$(u, u, d), (u, u, u)\}$$

where the notation is obvious. According to the ideas of Section 3, we are looking for non-path-dependent European style *SRA* and *GD*, so we can simplify the set of states of nature, which becomes

$$\Omega^* = \{\{(d, d, d)\}, \{(d, d, u), (d, u, d), (u, d, d)\}, \{(d, u, u), (u, d, u),$$
$$(u, u, d)\}, \{(u, u, u)\}\},$$

which may be identified with

$$\Omega^* = \{0, 1, 2, 3\},$$

associated with the four possible final pay-offs $y_0 = 25$, 100, 400, 1600. Equality $R = 1.01$ implies that $R_{y_0} > 1.01$ must hold to deal with a risk adverse market. Suppose that $R_{y_0} = 1.35$. Take $C = 100.0301$ as the initial price of the *SDA*. Expression (36) yields the *SDF* for every $\omega \in \Omega^*$, so (20) holds for $\mu_0 \leq 71.69\%$. For instance, we can apply the algorithm in Remark 3 for $\mu_0 = 70\% = 0.7$, and the obtained *SDA* final pay-off y^* is

$$y^* = 0, 231.89, 0, 0$$

[7] The authors sincerely thank "*WELZIA MANAGEMENT, SGIIC, S.A.*" for providing us with several database.

for $\omega = 0, 1, 2, 3$. Obviously, pay-off y^* cannot be replicated unless we recover the whole set Ω and the whole process (39). The trees below provide us with the stochastic evolution of both the price of y^* and its delta. Obviously, since y^* must be replicated with a self-financing combination of the risk-free asset and delta units of (39), the stochastic evolution of the investment in the risk-free asset may be obtained in a straightforward manner and may be omitted in this summary.

$$\left(\begin{array}{ccccccc}
\multicolumn{7}{c}{The_price_procees_for_the_SRA_y^*}\\
&&&&&&0\\
&&&&&\nearrow&\\
&&&&0&&\\
&&&\nearrow&&\searrow&\\
&&99.02&&&&0\\
&\nearrow&&\searrow&&\nearrow&\\
100&&&&151.53&&\\
&\searrow&&\nearrow&&\searrow&\\
&&102.02&&&&231.89\\
&&&\searrow&&\nearrow&\\
&&&&78.06&&\\
&&&&&\searrow&\\
&&&&&&0
\end{array}\right),
\left(\begin{array}{ccccc}
\multicolumn{5}{c}{Delta_of_y^*}\\
&&&&0\\
&&&\nearrow&\\
&&-0.252&&\\
&\nearrow&&\searrow&\\
-0.01&&&&-0.773\\
&\searrow&&\nearrow&\\
&&0.49&&\\
&&&\searrow&\\
&&&&3.092
\end{array}\right). \tag{40}$$

Finally, let us remark that

$$VaR_{0,7}(y^* + y_0 - \mathrm{IE}(y^* z_\pi)) = -228.86$$

and

$$CVaR_{0,7}(y^* + y_0 - \mathrm{IE}(y^* z_\pi)) = -145.62.$$

□

Example 2. Once we know the *SDA* of Example 1, let us build the *GD*. According to Remark 4, we can solve (33) and (34) for $C = 100$ and $m = 1000, 2000, 3000, \ldots$. If we select $m = 30{,}000$, then the solution of (34) leads to the final pay-off

$$x_m^* = 40924.5, 42124.5, 42424.5, 0,$$

and therefore

$$y_m^* = x_m^* - m = 10924.5, 12124.5, 12424.5, -30000. \tag{41}$$

The associated risks are

$$VaR_{0.7}(y_m^* + y_0 - \mathrm{IE}(y_m^* z_\pi)) = -12424.5$$

and

$$CVaR_{0.7}(y_m^* + y_0 - \mathrm{IE}(y_m^* z_\pi)) = -897.16.$$

Since, according to the first constraint in (33), the required investment at $t = 0$ is $C/R^3 = 97.06$, (8) and (11) imply that the expected return Y_m associated to y_m^* satisfies

$$\begin{aligned}
Y_m &= \frac{\mathrm{IE}(y_m^* + y_0)}{97.06} - 1 \geq \frac{-CVaR_{0.7}(y_m^* + y_0)}{97.06} - 1\\
&= \frac{-CVaR_{0.7}(y_m^* + y_0 - \mathrm{IE}(y_m^* z_\pi)) + \mathrm{IE}(y_m^* z_\pi)}{97.06} - 1\\
&= \frac{997.16}{97.06} - 1 = 9.2736 = 927.36\%.
\end{aligned}$$

As can be seen, the proposed *GD* (41) yields a significant expected return with a very negative *VaR* and *CVaR*. Obviously, the whole tree reflecting the stochastic evolution of this *GD* price and delta may be computed with the same method as that used in order to obtain (40). □

Example 3. Next let us assume that the parameters of the binomial model may be dynamic. For instance, in practice, the binomial model is an approximation of the Black and Scholes model. Thus, the volatility σ of the process may be dynamically estimated, and therefore

$$u = e^{\sigma\sqrt{\nabla(t)}} \tag{42}$$

and

$$d = e^{-\sigma\sqrt{\nabla(t)}} \tag{43}$$

must be re-adapted. Both algorithms in Remarks 3 and 4 may be re-applied at every trading date, so R, u and d may become different at a future trading date $\omega\nabla(t)$, $\omega = 1, 2, \ldots, n - 1$. The new *SRA* (or *GD*) may be re-calculated with the same price as that of the *SRA* we were holding. Hence, the *SRA* is still replicated with a self-financing combination of the risk-free and the risky asset.

In order to illustrate the method, consider again the parameters of Example 1. Suppose that after one period the underlying asset price becomes 400 (see (39)), $u = 4$ and $d = 0.25$. Then, prices and deltas of the *SRA* in (40) must me modified according to the new trees

$$\left(\begin{array}{ccccccc} \multicolumn{7}{l}{The\_price\_procees\_for\_the\_SRA\_y^{*}} \\ & & & & & & 0 \\ & & & & & \nearrow & \\ & & & & 235,97 & & \\ & & & & & \searrow & \\ & & & \nearrow & & & \\ & & 94.70 & & & & 298.91 \\ & & & \searrow & & & \\ & & & & & \nearrow & \\ 100 & \nearrow & & & 59.98 & & \\ & & & & & \searrow & \\ & & & & & & 0 \end{array}\right), \quad \left(\begin{array}{ccccc} \multicolumn{5}{c}{Delta\_of\_y^{*}} \\ & & & & -0.05 \\ & & & \nearrow & \\ & & 0.117 & & \\ & & & \searrow & \\ -0.01 & \nearrow & & & 0.8 \end{array}\right)$$

Optimal risk levels have become different too due to the evolution of u and d. The new values are $VaR_{0,7}(y^* + y_0 - \mathrm{IE}(y^* z_\pi)) = -602.31$ and $CVaR_{0,7}(y^* + y_0 - \mathrm{IE}(y^* z_\pi)) = -181.64$. Obviously, if one period later the parameters become different again, the right hand side of the trees above will have to be modified for the second time. □

Example 4. Let us construct a *SRA* for the *SP*500 American index. We have selected the year 2011 and the period [0, *T*] = [March_14th,March_18th] because the return of the *SRA* was very large, but, in general, our empirical test affecting the *SP*500 index and the whole year 2011 reveals that the *SRA* may generate very acceptable returns.

On March 14th, 2011, the *SP*500 index value was 1296.39, and the *VIX* index value was 21.13%. Suppose that *VIX* index may be interpreted as a predictor of the *SP*500 volatility. We will construct the *SRA* with current price 100 and maturity T = [March, 18th, 2011], by dealing with a binomial model such that $\nabla(t)$ equals one day (or 1/260 years), and the risk-free rate vanishes. Parameters u and d are given by (42) and (43). Obviously, there are four periods, so $\Omega^* = \{0, 1, 2, 3, 4\}$. If $\mu_0 = 70\%$ and we assume that the *SP*500 annual expected return equals 20%, then the algorithm of Remark 3 leads to the *SRA*

$$y^* = 0, 144.93, 111.39, 76.97, 41.63.$$

The final (March 18th) index value was 1279.2, whereas the final value of the *SRA* was 144.93, i.e., it generated an absolute profit equal to 44.93. As in the examples above, the evolution of the *SRA* price delta can be given too. □

5. Sensitivity analysis

Example 3 has illustrated that changes in the parameters of the pricing model within the period $(0, T)$ may be incorporated to the algorithms. Nevertheless, it may be interesting to have a previous estimation about the effect that those changes could provoke on the optimal risk value $CVaR_{\mu_0}(y^* + y_0 - \mathrm{IE}(y^* z_\pi))$.

This section is devoted to quantifying the effect on $CVaR_{\mu_0}(y^* + y_0 - \mathrm{IE}(y^* z_\pi))$ of measurement errors and changes in the pricing model. To this purpose we will draw on the classical "Envelope Theorem" of Mathematical Programming. Mainly, this theorem states that the optimal value sensitivity (partial derivative) with respect to every involved parameter equals a partial derivative of the Lagrangian Function.

Consider $0 < \mu_0 < 1$ and Problem (15) for a variable $C > 0$ belonging to an open set of IR and variables y_0 and z_π belonging to open sets of IR^Ω. Suppose that $\mathrm{IP}(z_\pi > 1/(1-\mu_0)) > 0$ holds for every feasible z_π. Theorem 2 guarantees the existence of solution for (15). Define $CVaR^*_{\mu_0}(C, y_0, z_\pi)$ as the optimal value of (15), that depends on (C, y_0, z_π).

Theorem 6. *Function $CVaR^*_{\mu_0}$ is Fréchet differentiable and*

$$\frac{\partial CVaR^*_{\mu_0}}{\partial C} = -\lambda^*, \quad \frac{\partial CVaR^*_{\mu_0}}{\partial y_0} = -z^* \text{ and } \frac{\partial CVaR^*_{\mu_0}}{\partial z_\pi} = (1+\lambda^*)y^*.$$

Proof. The Envelope Theorem of Mathematical Programming implies that $CVaR^*_{\mu_0}$ is Fréchet differentiable if so is the Lagrangian Function at the (*primal*, *dual*) solution, and both differentials coincide. The Lagrangian Function of (15) is (see Balbás et al., 2010c, for a general Lagrangian Function of optimization problems involving risk measures)

$$\mathcal{L}(y^*, \lambda^*, z^*, C, y_0, z_\pi) = -\lambda^* C - \mathrm{IE}(y_0 z^*) + \mathrm{IE}(y^*(1+\lambda^*)z_\pi - z^*),$$

and the conclusion of the theorem trivially follows. □

Remark 5. Theorem 6 enables us to give an approximation of the optimal risk level variation $\nabla(CVaR^*_{\mu_0})$ with respect to modifications of the parameters. In particular,

$$\nabla(CVaR^*_{\mu_0}) \approx -\lambda^* \nabla(C) - \mathrm{IE}(z^* \nabla(y_0)) + (1+\lambda^*)\mathrm{IE}(y^* \nabla(z_\pi)). \tag{44}$$

We know that y^* vanishes outside $\omega_0 < \omega < \omega_1$ (Theorem 4) so there is no sensitivity with respect to errors of the *SDF* estimates unless they significantly affect the central values of Ω^*. This is important because two different pricing models usually reflect significant differences on the distribution tails (heavy tails), rather than the central values. Besides, the sensitivity with respect to the pay-off y_0 becomes important if errors arise for small values of $\omega \in \Omega^*$, since in such a case $z^*(\omega) = 1/(1-\mu_0)$ (Theorem 4e). This sensitivity is negligible for high values of $\omega \in \Omega^*$. □

Remark 6. In the particular case of the binomial model, if u and d are modified, (35) and (36) obviously lead to

$$\mathrm{IE}(z^*\nabla(y_0)) \approx W\left[\sum_{\omega=0}^{n}\omega u^{\omega-1}d^{n-\omega}z^*(\omega)\mathrm{IP}(\omega)\right]\nabla(u)$$

$$+W\left[\sum_{\omega=0}^{n}(n-\omega)u^{\omega}d^{n-\omega-1}z^*(\omega)\mathrm{IP}(\omega)\right]\nabla(d) \quad (45)$$

and

$$\mathrm{IE}(y^*\nabla(z_\pi)) \approx \left[\sum_{\omega=0}^{n}(n-\omega)\left(\frac{R-d}{R_{y_0}-d}\right)^{\omega}\left(\frac{u-R}{u-R_{y_0}}\right)^{n-\omega-1}\right.$$

$$\left.\frac{R-R_{y_0}}{(u-R_{y_0})^2}y^*(\omega)\mathrm{IP}(\omega)\right]\nabla(u)+\left[\sum_{\omega=0}^{n}\omega\left(\frac{R-d}{R_{y_0}-d}\right)^{\omega-1}\right.$$

$$\left.\left(\frac{u-R}{u-R_{y_0}}\right)^{n-\omega}\frac{R-R_{y_0}}{(d-R_{y_0})^2}y^*(\omega)\mathrm{IP}(\omega)\right]\nabla(d). \quad (46)$$

Therefore, (44)–(46) will give the variation $\nabla(CVaR^*_{\mu_0})$ of the optimal risk level with respect to the parameters u and d. Similarly, bearing in mind (35) and (36), one can compute closed formulas of the sensitivity with respect to the "risk-free rate" R and the risky expected return R_{y_0}. Finally, if we take the binomial model as an approximation of the Geometric Brownian Motion and therefore (42) and (43) provide u and d as functions of the volatility σ of the risky asset, then

$$\nabla(u) \approx \sqrt{\nabla(t)}e^{\sigma\sqrt{\nabla(t)}}\nabla(\sigma)$$

and

$$\nabla(d) \approx -\sqrt{\nabla(t)}e^{-\sigma\sqrt{\nabla(t)}}\nabla(\sigma),$$

and (44)–(46) trivially lead to expressions providing us with the sensitivity of $CVaR^*_{\mu_0}$ with respect to the volatility of the underlying asset. □

6. Conclusions

This paper has given practical algorithms allowing us to compute shadow riskless assets and good deals for discrete time arbitrage-free complete pricing models. The interest seems to be clear. Indeed, shadow riskless assets permit managers to reduce the level of capital requirements, whereas good deals permit investors to outperform every alternative strategy if the performance criteria are expected return and risk. Needless to say that return/risk ratios are crucial in order to rank the effectiveness of portfolio managers.

The given algorithms are general since they apply under weak assumptions about the pricing model. Essentially, the existence of good deal is the unique required condition. Nevertheless, for illustrative reasons, numerical experiments have been given for the binomial model. One of the numerical experiments has been built with real American data.

The algorithms allow us to incorporate changes of the model parameters in a dynamic setting. These changes may be caused by both evolutions of the market conditions and/or measurement errors. Nevertheless, it may be also interesting to have a previous information about the possible effect of these changes. The sensitivity of the solutions with respect to important elements has been given. Among other interesting elements, one can consider the pricing rule (or the stochastic discount factor) or the manager's random final pay-off.

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