

Article

Debt refinancing and credit risk

Santiago Forte^{a,*}, Juan Ignacio Peña^b^a ESADE Business School, Ramon Llull University, Barcelona, Spain^b Universidad Carlos III de Madrid, Spain

ARTICLE INFO

Article history:

Received 24 July 2009

Accepted 8 October 2010

JEL classification:

G12

G13

G32

Keywords:

Refinancing contract

Credit rating

Credit spreads

ABSTRACT

Many firms choose to refinance their debt. We investigate the long run effects of this extended practice on credit ratings and credit spreads. We find that debt refinancing generates systematic rating downgrades unless a minimum firm value growth is observed. Deviations from this growth path imply asymmetric results. A lower firm value growth generates downgrades and a higher firm value growth generates upgrades, as expected. However, downgrades tend to be higher in absolute terms. We also find that the inverse relation between credit spreads and risk free rate that structural models usually predict still holds in this setting, but only in the short run. This negative relation will turn to be null in the medium run and positive in the long run.

© 2009 Asociación Española de Finanzas. Published by Elsevier España, S.L. All rights reserved.

1. Introduction

Many different reasons may be behind the decision of a particular firm to issue new debt: financing a new investment project, getting funds to operate in a period of low earnings, or simply refinancing existing debt. The purpose of the issue is not irrelevant. An example is provided by Gande et al. (1997), who examine differences in debt securities underwritten by Section 20 subsidiaries of bank holding companies relative to those underwritten by investment houses. Among other results, they find that when debt is used to refinance existing debt, the credit spread is on average 14 basis points above the one that results considering “other purposes”. Intuitively, if the purpose of the issue is to finance a new investment project that will increase the expected earnings of the firm, and its market value, then the risk premium should be lower than in the case in which debt proceeds are used to refinance existing debt, because in this situation no added value is created.¹ Refinancing

current debt, however, seems to be one of the most important – if not the first – reason to issue new debt. The mentioned article for instance considers a sample in which 43.5% of the issues had the purpose to refinance existing debt. More evidence in this line is given by Hansen and Crutchley (1990), who investigate the relationship between corporate earnings and sales of common stocks, convertible bonds, and straight bonds. In this case, 64% of straight bond issues were used at least partially to refinance existing debt. This ratio grows up to 72% when they consider convertible debt.

In spite of the fact that debt refinancing appears as an extending practice, we know little about how this can potentially affect the credit standing of a firm in the long run. The present article represents a first attempt in this direction. We introduce the concept of refinancing contract, modeling dividend rates, maturities, and nominal debt payments, as part of this contract. We then describe the credit spreads faced by the firm to refinance as a function of the firm characteristics and the specific contract selected, and analyze how the fact that firms choose to refinance their debt can potentially affect the credit rating and the credit spreads of those firms in the long run.

The main conclusions of the paper are the following. First, debt refinancing generates systematic credit rating downgrades unless a minimum firm value growth is observed. Deviations from such a firm value growth path imply asymmetric results. While a lower firm value growth results in systematic downgrades and a higher firm value growth in systematic upgrades, as expected, the same deviation will have a higher effect in absolute terms when it is negative than when it is positive. Said in other words, we should expect rating migrations to exhibit a certain degree of inertia among those

* Corresponding author.

E-mail addresses: santiago.forte@esade.edu (S. Forte), ypenya@eco.uc3m.es (J.I. Peña).

¹ The theoretical argument given by Gande et al. (1997) to justify different credit spreads depending on the purpose of the issue is nevertheless not the same we provide here. They argue that when a firm has a bank loan, and wants to refinance it with public debt, potential buyers may expect that the firm has been induced by the bank to take this decision because the loan is at risk. However, significant differences are found even when new issues are classified as “investment grade”. This indicates that debt refinancing is not a practice that firms use only in case of, or to avoid, credit distress.

companies that choose to refinance their debt, and this inertia should be stronger in the case of downgrades than in the case of upgrades. Evidence in this regard has been actually provided by Altman and Kao (1992). Specifically, they find positive autocorrelation in S&P downgrades and upgrades for high-yield bonds, being this autocorrelation stronger in the case of downgrades.

The second main conclusion of the paper is that the traditional prediction of an inverse relation between credit spreads and risk free rate (Merton, 1974; Leland, 1994; Longstaff and Schwartz, 1995) holds just in the short run. Debt refinancing makes such relation to turn null in the medium run and positive in the long run. Evidence of this dynamic relation is in fact provided by Longstaff and Schwartz (1995) (negative effect in the short run) and Guha et al. (2001) (positive effect in the long run). Overall, we conclude that the common practice of debt refinancing and results provided in this paper allow explaining, in a unified framework, the empirical evidence on the dynamic relation between risk free rate and credit spreads.

The rest of the article is organized as follows: Section 2 introduces the concept of refinancing contract, and describes when, and how, a contract of this type with an arbitrary number of future payment dates n , can be designed. Section 3 analyzes the effects of debt refinancing under the specific cases of $n = 1$ and $n = 2$.² Finally Section 4 summarizes the main findings of the paper.

2. The general case

The following assumption summarizes our theoretical framework.

Assumption A.

A1: There are no taxes, problems concerning indivisibility, bankruptcy costs, transactions costs, or agency costs.

A2: Trading takes place continuously.

A3: There exists a risk free asset with constant interest rate r , that applies for borrowing and lending, and for any maturity.

A4: Every individual acts as if she can buy or sell as much of any security as she wishes without affecting the market price.

A5: Individuals may take short positions in any security, including the risk free asset, and receive the proceeds of the sale. Restitution is required for payouts made to securities held short.

A6: Modigliani–Miller Theorem obtains, that is, the firm value is independent of its capital structure.

A7: The firm value, V , follows the diffusion process given by

$$dV = (\mu - \delta)V dt + \sigma V dz, \quad (1)$$

where μ is the expected rate of return on V , δ is the constant rate of firm value which is paid to equity holders as dividends, σ is the volatility of the rate of return which will be assumed to be constant, and z is a standard Brownian motion.

No assumption is made at this moment about the profile of nominal payments that constitute the corporate debt. We simply assume that a debt contract was signed at some period prior to current period t . Under this contract, at least a certain debt payment has to be satisfied at some future period $\tau > t$. This, and any posterior debt payment, is to be financed by issuing additional equity. Under these conditions the equity and debt values will be a function of the firm value and time. Denote then the equity value as $S(V, t)$, and the debt value as $F(V, t)$. We start by defining the general form of any refinancing contract.

Definition. A refinancing contract between the firm and the debt holders at τ , is a vector $\Theta \equiv (\delta, \Psi, \mathcal{Y}) \in \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n$, with $n < \infty$, by which:

- The firm, which is assumed to maximize equity holders' wealth, promises (under limited liability) the payment of Ψ at \mathcal{Y} , that is, the payment of ψ_i at τ_i , where $\psi_i \in \Psi$, $\tau_i \in \mathcal{Y}$, $i = 1, \dots, n$, and $\tau_1 > \tau$.
- The firm also restricts itself to apply a dividend rate equal to δ , and loses the right to issue new debt. These restrictions apply until Θ has been canceled, either by satisfying nominal payments regularly (issuing new equity), or by means of a posterior debt refinancing contract.
- The debt holders renounce to $F(V, \tau)$.

We say that Θ is feasible, if and only if the firm and the debt holders are willing to sign Θ . The set of feasible Θ is denoted by Θ^F .

A refinancing contract (RC) is therefore similar to a standard debt contract. The main difference is that debt holders do not provide cash to the firm at issuance, but the renounce to the payment of current debt (covenant c). In addition, we include an agreement on dividends (covenant b). This agreement prevents equity holders from extracting a higher share of the firm value (with the implied reduction for debt holders), by increasing the dividend rate after signing the contract.

The following lemma establishes a necessary condition for a feasible set of refinancing contracts to exist.

Lemma. Let $S(V, \Theta, \tau)$ and $F(V, \Theta, \tau)$ denote the equity and debt value at τ when the value of the firm is V , the debt profile consists on the payment of Ψ at \mathcal{Y} , and the dividend rate is δ . Then, $\Theta \in \Theta^F$ if and only if $S(V, \Theta, \tau) = S(V, \tau)$, implying $S(V, \tau) > 0$ as a necessary condition for a feasible Θ to exist.

Proof. See Appendix A.

Although a formal statement of the proof is in the appendix, the intuition is straightforward. Modigliani and Miller's Theorem implies that no value is created or destroyed in the firm by refinancing its debt. As a consequence, equity holders can neither gain, nor lose due to refinancing. If they are worst off with the contract they will simply refuse it, but if they are better of this means that debt holders are worst off, and in this case they will be those who refuse the contract. This allows us to identify the set of feasible refinancing contracts with the set of contracts that leave equity holders with the same value. On the other hand, limited liability makes the equity value to be strictly positive if no current debt payment has to be satisfied, which is the case after signing the contract. This makes $S(V, \tau) > 0$ finally to be a necessary condition for a feasible contract to exist. One implication is that equity and debt can still be valued assuming that debt will be paid by issuing additional equity. The reason is that the possibility of refinancing will not alter their welfare with respect to this situation in any sense. We set up this argument as follows:

Remark. Refinancing does not alter neither equity holders, nor debt holders' wealth. This implies that $S(V, \Theta, \tau)$ and $F(V, \Theta, \tau)$ can be valued assuming that debt payments are to be financed by issuing new equity, even if this never happens, that is, even if the firm always chooses to refinance its debt.

Searching for a feasible contract implies at this point searching for $2n + 1$ elements. The following restriction on the relation between debt payments, and on the time spread between these payments, will allow us to reduce the dimension of the problem to 3.

Restriction. Let $\Psi = \psi_1 \Phi$, where Φ is the n -dimensional vector which first element ϕ_1 equals 1, and the remaining are some

² This last case can be seen as a simplification to short and long term debt.

fixed values $\phi_i > 0 \forall i \geq 2$. Let also $\Pi = (\tau_1 - \tau)\Lambda$, where Π denotes the n -dimensional vector which first element π_1 equals $(\tau_1 - \tau)$, and π_i equals $(\tau_i - \tau_{i-1}) \forall i \geq 2$. Λ on the other hand denotes the n -dimensional vector which first element η_1 equals 1, and the remaining are some fixed values $\eta_i > 0 \forall i \geq 2$.

If we denote $\Delta \equiv (n, \Phi, \Lambda)$ the vector that describes a particular corporate debt structure, then, for any Δ , $(\delta, \Psi, \mathcal{T}) \in \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n$ is in fact fully described by $(\delta, \psi_1, \tau_1) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}$. The following assumption describes the behavior of the equity value as a function of these three elements. Although this behavior is stated as an assumption, it will be shown to hold later on for the specific cases of $n = 1$ and $n = 2$.

Assumption B (For any given Δ).

B1: $S(V, \Theta, \tau)$ is a continuous and strictly decreasing function in ψ_1 (CSD (ψ_1)), with $S(V, \Theta, \tau)|_{\psi_1=0} = V$, and $\lim_{\psi_1 \rightarrow \infty} S(V, \Theta, \tau) = E^{R.N.} \int_{\tau}^{\tau_1} \delta V(s) e^{-r(s-\tau)} ds = V(1 - e^{-\delta T_1})$.³

B2: $S(V, \Theta, \tau)$ is a continuous and strictly increasing function in τ_1 (CSI (τ_1)), with $\lim_{\tau_1 \rightarrow \tau} S(V, \Theta, \tau) = \text{Max}\{0, V - \sum_{i=1}^n \psi_i\}$,⁴ and $\lim_{\tau_1 \rightarrow \infty} S(V, \Theta, \tau) = V$. Denote $\hat{\psi}_1$ the ψ_1 value such that $S(V, \tau) = V - \psi_1 \sum_{i=1}^n \phi_i$, that is, $\hat{\psi}_1 = F(V, \tau) / \sum_{i=1}^n \phi_i$.

B3: $S(V, \Theta, \tau)$ is a continuous and strictly increasing function in δ (CSI (δ)), with $\lim_{\delta \rightarrow \infty} S(V, \Theta, \tau) = V$.

Assumption B1 asserts that the equity value is a continuous and strictly decreasing function in the nominal payments that equity holders have to satisfy. $S(V, \Theta, \tau)|_{\psi_1=0} = V$ recognizes that if there is no debt, then the equity holders own the firm. $\lim_{\psi_1 \rightarrow \infty} S(V, \Theta, \tau) = E^{R.N.} \int_{\tau}^{\tau_1} \delta V(s) e^{-r(s-\tau)} ds$ indicates that as nominal debt tends to infinity, default at τ_1 becomes unavoidable, and the unique value associated to equity is the value of the dividends that will be received until the first payment is required. Standard arguments allow us to use risk neutral valuation. Assumption B2 indicates that as τ_1 tends to τ , new corporate debt tends to consist in a single payment satisfied at τ . As τ_1 tends to infinity, the present value of future debt payments, $F(V, \Theta, \tau)$, tends to zero, and the equity value tends to the firm value. Finally Assumption B3 states that $\lim_{\delta \rightarrow \infty} S(V, \Theta, \tau) = V$, which reflects that, for any $\tau_1 > \tau$, in the limit case of $\delta = \infty$ the equity holders liquidate the firm before any debt payment can be required. Note also that $S(V, \Theta, \tau)|_{\delta=0}$ coincides with the case presented by Geske (1977).

All of the above allows us to establish the following theorem.

Theorem. Suppose $S(V, \tau) > 0$, and let $\varphi^\delta \equiv \mathbb{R}_+$, $\varphi^{\psi_1} \equiv (\hat{\psi}_1, \infty)$, $\varphi^{\tau_1} \equiv (\tau, \infty)$. Consider any sequence $\alpha - \beta - \gamma$, where α is chosen in φ^α , and define $\varphi^{\beta|\alpha}$ as the subset of φ^β for which $S(V, \Theta, \tau) = S(V, \tau)$ reaches a solution for at least one $\gamma \in \varphi^\gamma$, given α . For any Δ , $\varphi^{\beta|\alpha}$ is a non empty set. Moreover, for any $\alpha \in \varphi^\alpha$, and $\beta \in \varphi^{\beta|\alpha}$, there is only one $\gamma \in \varphi^\gamma$ such that $S(V, \Theta, \tau) = S(V, \tau)$.

Proof. See Appendix A.

Corollary 1. $\Theta^F \neq \emptyset$ if and only if $S(V, \tau) > 0$.

Proof. This holds given lemma and theorem above.

The theorem asserts that, whenever $S(V, \tau) > 0$, a feasible RC with any arbitrary capital structure can be generated, and also describes how can it be constructed. Although this feasible Θ is not unique for a given Δ , and there is actually an infinite number of elements

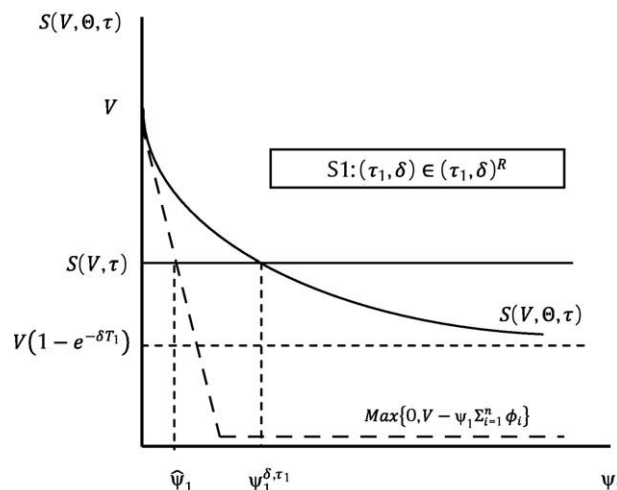


Fig. 1. This figure shows that for any pair of values $(\tau_1, \delta) \in (\tau_1, \delta)^R$ there exists one, and only one $\psi_1 \in \varphi^{\psi_1}$, such that the RC generated in this way is feasible.

in $\Theta^F | \Delta$, not everything is possible. Choosing one element in $\Theta^F | \Delta$ could be seen as a matter of priority. Take for instance the sequence $\tau_1 - \delta - \psi_1$. The maturity of the first payment, τ_1 , is freely chosen in the interval (τ, ∞) . However, this election restricts the range of the dividend rates, δ , that can be selected in $[0, \infty)$, and the election of the dividend rate in the restricted interval finally determines a unique first debt payment, ψ_1 , in $(\hat{\psi}_1, \infty)$. On the other hand, Corollary 1 implies that refinancing is feasible under the same conditions that it is feasible to issue new equity to pay the debt. Note also that $S(V, t) > 0 \forall t < \tau$ given limited liability, what would allow the firm to refinance for any $t < \tau$.

We end this section with an additional corollary which establishes a joint restriction on the possible values of (τ_1, δ) , (τ_1, ψ_1) , and (δ, ψ_1) , that need to be satisfied in any refinancing contract. This corollary follows directly from the proof of the theorem.

Corollary 2. Consider the following scenarios:

- S1:** $(\tau_1, \delta) \in (\tau_1, \delta)^R$, where $(\tau_1, \delta)^R \equiv (\tau_1, \delta) \in \varphi^{\tau_1} \times \varphi^\delta | \delta T_1 < \ln[V/F(V, \tau)]$;
- S2:** $(\tau_1, \psi_1) \in (\tau_1, \psi_1)^R$, where $(\tau_1, \psi_1)^R \equiv (\tau_1, \psi_1) \in \varphi^{\tau_1} \times \varphi^{\psi_1} | S(V, \Theta, \tau)|_{\delta=0} \leq S(V, \tau)$;
- S3:** $(\delta, \psi_1) \in (\delta, \psi_1)^R$, where $(\delta, \psi_1)^R \equiv (\delta, \psi_1) \in \varphi^\delta \times \varphi^{\psi_1}$.

For any scenario $(\alpha, \beta) \in (\alpha, \beta)^R$, there is one, and only one $\gamma \in \varphi^\gamma$, such that the RC generated in this way is feasible.

Proof. See Appendix A.

Figs. 1–3 reflect the arguments in Corollary 2 for the three possible scenarios: S1, S2 and S3.

The fact that any possible debt structure can be chosen, joint with the freedom to select any sequence $\alpha - \beta - \gamma$, gives the RC the possibility to be as “imaginative” as one may desire. We may, nevertheless, describe how to design two types of contracts that are commonly used in practice: loans which are paid in equal monthly installments, and coupon bonds.

In our notation, the type of loan described can be represented by a vector Δ , where n is equal to the number of years times 12, and Φ as well as Λ are given by a vector of dimension n with all elements equal to 1. In order to guarantee that we obtain monthly payments, we may start by choosing τ_1 . If we fix τ equal to zero, and if we assume that μ and σ are in annual terms, then τ_1 equal to $1/12$ generates the monthly payments desired. Corollary 2 finally ensures that for any δ lower than $12 \times \ln[V/F(V, 0)]$ we will be able to find a ψ_1 value in the interval $(F(V, 0)/n, \infty)$, such that the resulting

³ $T_1 = \tau_1 - \tau$.

⁴ $V - \sum_{i=1}^n \psi_i = V - \psi_1 \sum_{i=1}^n \phi_i$.

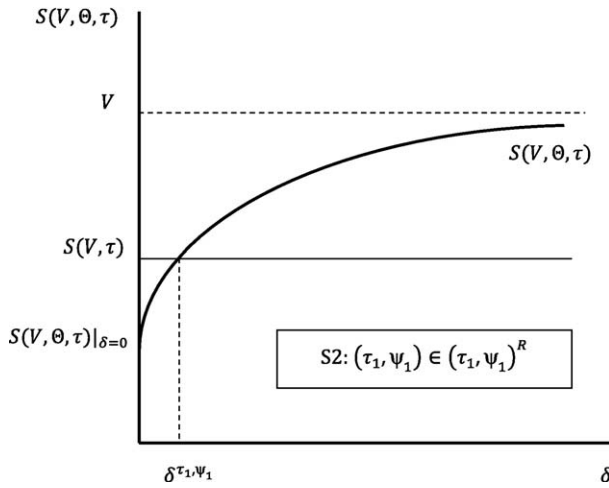


Fig. 2. This figure shows that for any pair of values $(\tau_1, \psi_1) \in (\tau_1, \psi_1)^R$ there exists one, and only one $\delta \in \varphi^\delta$, such that the RC generated in this way is feasible.

contract allows the firm to refinance; a contract that implies equal monthly installments.

A coupon bond will require a little bit more of elaboration. Clearly, this way to refinance needs the *debt principal* to be equal to $F(V, \tau)$. If coupons are paid annually, then n will be the number of years, and Λ will be a vector of ones of dimension n . Φ this time will be given by a vector of dimension n , with all elements equal to 1 but the element n , which will be equal to $(1 + p)$. Consider again period τ equals zero, and choose $\tau_1 = 1$ to guarantee annual payments. Again we can use Corollary 2 to ensure that there exists a ψ_1 in the interval $(F(V, 0)/(n + p), \infty)$ that allows the firm to refinance for any δ lower than $\ln[V/F(V, 0)]$. According to Φ , we will have equal coupon payments between periods 1 and $n - 1$, and the payment of coupon plus $\psi_1 p$ at the final date n . But we need a pair of values (ψ_1, p) such that the equality $\psi_1 p = F(V, 0)$ holds. As a first step we may prove that such a pair exists. Note that this equality, joint with restriction $\psi_1 > F(V, 0)/(n + p)$, leads to $\psi_1 n > 0$. In short, equity holders could pay $F(V, 0)$ at $\tau = 0$, or defer this payment to a future period. Condition $\psi_1 n > 0$ simply says that at least a coupon payment is needed in order to charge the interests that debt holders will require for this deferment. The problem is that, in general, $\psi_1 p$ will not be equal to $F(V, 0)$. We could however use a tivial “trial

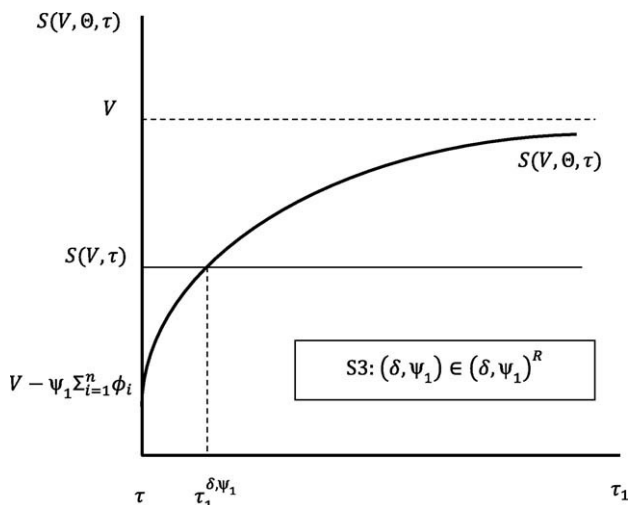


Fig. 3. This figure shows that for any pair of values $(\delta, \psi_1) \in (\delta, \psi_1)^R$ there exists one, and only one $\tau_1 \in \varphi^{\tau_1}$, such that the RC generated in this way is feasible.

and error” method to find these two values.⁵ At the end we will have designed a coupon bond that allows the firm to refinance.

We next analyze the specific cases of $n = 1$ and $n = 2$. These will be useful to provide the basic implications of the refinancing strategy.

3. Particular cases

3.1. $n = 1$

Although any possible initial debt structure could be considered, we will assume in this case that n remains constant along time. This means that a single zero coupon bond, with nominal ψ and maturity at τ , is replaced by a single zero coupon bond, with some nominal ψ_1 and some maturity $\tau_1 > \tau$, whenever $S(V, \tau) > 0$. In order to show that a feasible RC exists, we need to describe how $S(V, \Theta, \tau)$ is to be valued. Specifically, we need to find the equity value at τ , when the corporate debt consists in the payment of ψ_1 at $\tau_1 > \tau$, and the dividend rate is δ , that is, $S(V, \Theta, \tau)$ for $\Theta \equiv (\delta, \psi_1, \tau_1)$.

$S(V, \Theta, \tau)$ has two sources of value. On one hand the value associated to the dividends that will be received between τ and τ_1 , $D(V, \tau)$. On the other hand the option value, $O(V, \tau)$, that comes from the possibility of acquiring the firm at τ_1 by paying ψ_1 . Applying risk neutral valuation we find that

$$D(V, \tau) = V(1 - e^{-\delta T_1}),$$

and⁶

$$O(V, \tau) = Ve^{-\delta T_1} N(d_1) - \psi_1 e^{-r T_1} N(d_2),$$

where

$$d_1 = \frac{\ln(V/\psi_1) + (r - \delta + (\sigma^2/2))T_1}{\sigma\sqrt{T_1}},$$

$$d_2 = d_1 - \sigma\sqrt{T_1}.$$

Finally,

$$S(V, \Theta, \tau) = V(1 - e^{-\delta T_1}) + Ve^{-\delta T_1} N(d_1) - \psi_1 e^{-r T_1} N(d_2), \quad (2)$$

where it is straightforward to check that Assumption B actually holds.⁷

Fig. 4 describes two of the six alternative ways to design a RC with $n = 1$: those associated to the sequences $\tau_1 - \delta - \psi_1$ and $\tau_1 - \psi_1 - \delta$, respectively.⁸

Consider first $\tau_1 - \delta - \psi_1$. In principle, any dividend rate between zero and infinity would be feasible in a RC. However, for a given maturity τ_1 strictly greater than the refinancing period τ , the set of possible dividend rates reduces to the interval $[0, \delta_2)$. We may say that the higher the debt maturity, the lower the dividend rate that debt holders will be willing to accept to refinance existing debt. Now let us assume that a debt maturity has been chosen. Fig. 4 makes clear that the higher δ , the higher the debt payment, or in other words, the higher the yield that debt holders will charge to the firm to refinance its debt.

Sequence $\tau_1 - \psi_1 - \delta$ can also be analyzed using Fig. 4. In this specific case of $n = 1$, it is clear that $\hat{\psi}_1 = \psi$. Nevertheless, any debt maturity τ_1 strictly greater than the refinancing moment τ will lead to a strictly positive yield reflected in a higher ψ_1 . For a fixed maturity we observe on the other hand that the higher the yield charged

⁵ Guess p_0 , and evaluate $\psi_1(p_0)$. If $\psi_1 p_0 < F(V, 0)$, guess a higher p_1 ; if $\psi_1 p_0 > F(V, 0)$ guess a lower p_1 , and repeat up to the point in which $\psi_1 p_0 = F(V, 0)$.

⁶ $N(\cdot)$ denotes the standard normal cumulative distribution function.

⁷ A formal proof is available upon request, however.

⁸ $S(V, \Theta, \tau)_{\psi_1} > 0$, implying that $S(V, \Theta, \tau)$ is a strictly convex function in ψ_1 .

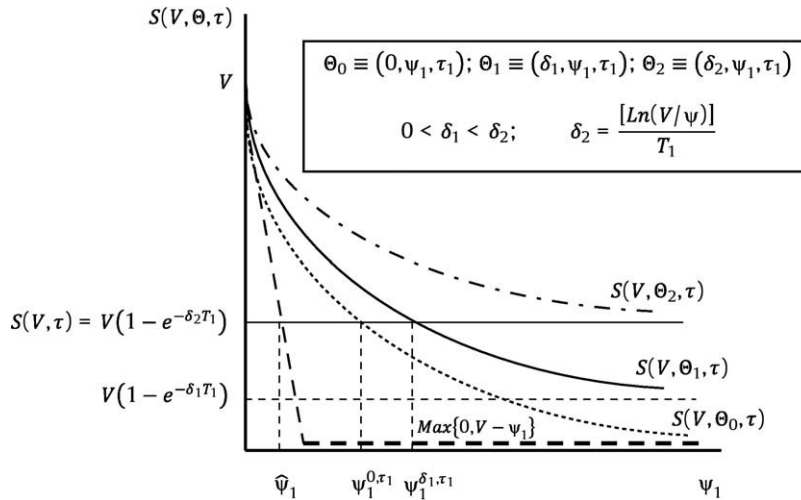


Fig. 4. This figure reflects the selection of a refinancing contract according to sequences $\tau_1 - \delta - \psi_1$ and $\tau_1 - \psi_1 - \delta$ for $n = 1$.

to equity holders, that is, the higher ψ_1 , the higher also the dividend rate. In short, equity holders demand a higher dividend rate as compensation for bearing a higher interest rate.

An interesting aspect is that, in fact, the credit spread (C.S) on corporate debt that results from refinancing will depend on the specific RC chosen in the feasible set; a feasible set that at the same time depends on the current firm value, the current nominal debt, the firm return volatility, and the risk free interest rate. We know that for any element Θ of this set, the new equity value, $S(V, \Theta, \tau)$, equals the previous one, $S(V, \tau)$. This can be expressed as $V - \psi_1 e^{-R\tau_1} = V - \psi$, where R is the interest rate associated to the new corporate debt. Then it is straightforward to show that

$$C.S = \frac{\ln(\psi_1/\psi)}{T_1} - r.$$

Although the dividend rate does not explicitly appear in the expression above, it does through its influence on ψ_1 and T_1 .

Given the firm characteristics and the risk free rate, that is, given a vector (V, ψ, σ, r) , the C.S will be a function of the specific contract chosen, which we represent by a vector (δ, ψ_1, τ_1) . We have seen however that only two of these three elements are “freely” chosen. Consider for instance (τ_1, δ) are selected according to the restriction imposed by Corollary 2 (S1), then the credit spread will be a function of $(V, \psi, \sigma, r, \delta, \tau_1)$, with $\delta T_1 < \ln(V/\psi)$. In order to make some comparative statics with respect to the C.S, we need to derive how ψ_1 depends on these variables and parameters. Let us define $\Gamma(V, \Theta, \tau) = S(V, \Theta, \tau) - S(V, \tau)$; then, Θ belongs to the feasible set if and only if $\Gamma(V, \Theta, \tau) = 0$, and the derivative of ψ_1 with respect to variable or parameter j will be given by

$$(\psi_1)_j = - \frac{\Gamma(V, \Theta, \tau)_j}{\Gamma(V, \Theta, \tau)_{\psi_1}},$$

what leads to $(\psi_1)_V < 0$, $(\psi_1)_\psi > 0$, $(\psi_1)_\sigma > 0$, $(\psi_1)_r > 0$, $(\psi_1)_\delta > 0$, $(\psi_1)_{\tau_1} > 0$.⁹ We may now describe the dependence of the C.S on the firm value and on the nominal debt to be refinanced.

$$C.S_V = \frac{(\psi_1)_V}{\psi_1 T_1} < 0,$$

$$C.S_\psi = \frac{(\psi_1)_\psi \psi - \psi_1}{\psi_1 \psi T_1} > 0.$$

A higher leverage ratio (a lower V or a higher ψ), means a higher credit risk, and the result is a higher C.S faced by the firm to refinance.

On the other hand,

$$C.S_\sigma = \frac{(\psi_1)_\sigma}{\psi_1 T_1} > 0.$$

It is also reasonable to observe that the higher the firm risk, the higher the credit spread on the firm debt.

The credit spread shows however to be independent of the risk free interest rate:

$$C.S_r = 0.$$

This result is not in contradiction with Merton’s (1974) prediction of an inverse relation between the credit spread of a zero coupon bond and the risk free rate. On the contrary, it extends the analysis to what will tend to happen in the medium and long run. To see this, assume an increase in the risk free rate at current period t . According to the arguments in Merton (1974) we will see an immediate fall in the credit spread at t ; however, previous arguments imply that this effect will only persist up to the refinancing date τ . In addition, the nominal of the new debt, ψ_1 , will be higher the higher the risk free rate. As a consequence, the credit risk premium that the firm will assume at τ_1 to refinance ψ_1 , will tend to be higher the higher the risk free rate at τ , given that this premium is an increasing function of the initial debt. We then conclude that under the assumption of debt refinancing with $n = 1$, an increase in the risk free rate reduces the credit spread in the very short run, has no effect in the medium run, and turns to have a positive effect in the long run. There is evidence at least of the first prediction. In fact, Longstaff and Schwartz (1995) show empirically that credit spreads exhibit an inverse relation with respect to changes in the interest rate on Government Bonds. It would be interesting to check whether the predictions for the medium and long run also hold.

The dependence of the credit spread on the dividend rate is described by

$$C.S_\delta = \frac{(\psi_1)_\delta}{\psi_1 T_1} > 0.$$

A higher dividend rate implies a lower expected firm value growth and a higher default probability, what leads to a higher credit spread.

⁹ Details available upon request.

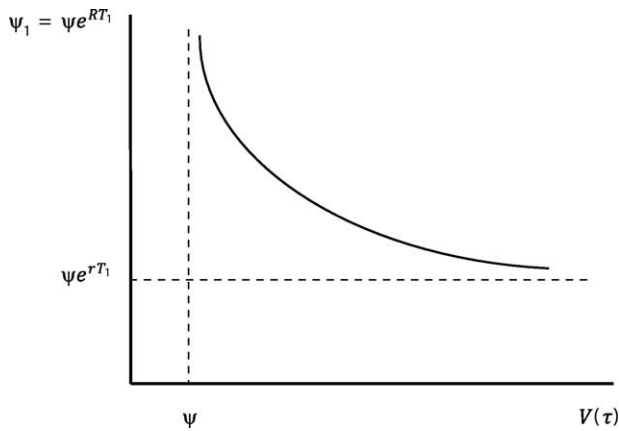


Fig. 5. This figure represents ψ_1 as a function of $V(\tau)$.

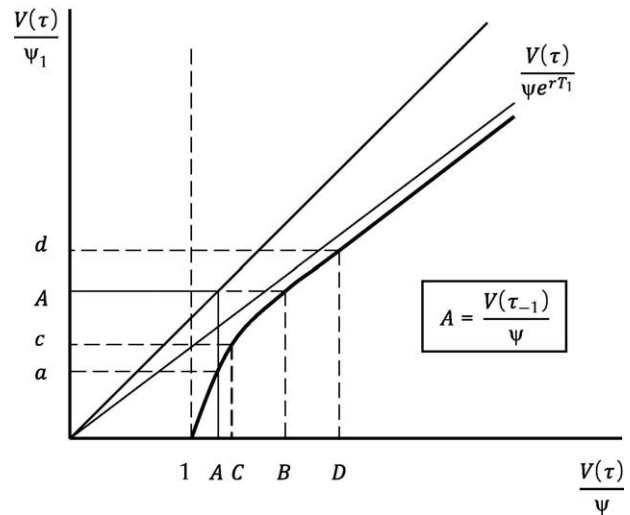


Fig. 6. This figure represents $V(\tau)/\psi_1$ as a function of $V(\tau)/\psi$.

Finally,

$$C.S_{\tau_1} = \frac{(\psi_1)_{\tau_1} - \psi_1 R}{\psi_1 T_1}$$

While an analytical solution for the sign of $C.S_{\tau_1}$ cannot be provided, it has shown to be positive in all simulations performed. Again, this result may seem inconsistent with the results in Merton (1974). It must be pointed out the substantial difference in the analysis of the time dependence followed by Merton and the one we drive here (not only the inclusion of a dividend rate). In fact, he sets the so-called “quasi debt-to-firm value ratio” constant. In order to keep this ratio equal to a fixed q for a given firm value and interest rate, ψ_1 should be determined as qVe^{rT} . In our case, however, we impose that the implied ψ_1 value is consistent with a feasible RC, what makes the ratio q to move from values below 1 to values above 1 for different maturities.¹⁰

The fact that a firm chooses to refinance its debt has also several implications in terms of the future evolution of credit ratings and credit spreads. To start with, we may represent the new nominal debt payment that results from refinancing at τ as a function of the firm value. This is show in Fig. 5. As the firm value tends to the default point ψ , the new nominal payment (and the credit spread that the firm has to face to refinance) tends to infinity. As the firm value tends to infinity on the other hand, the new payment tends to the previous payment capitalized at the risk free interest rate.¹¹

Let us now assume that the credit rating of the firm is measured at any time the firm refinances, as the ratio current firm value to new nominal debt. Fig. 6 follows directly from Fig. 5, and represents the credit rating at τ , $V(\tau)/\psi_1$, as a function of the ratio $V(\tau)/\psi$. Let $A = V(\tau_{-1})/\psi$ be the credit rating of the firm at issuance of ψ . Then, if the firm value stays constant between τ_{-1} and τ , the credit rating falls to a . The fact that the firm refinances makes possible to observe a downgrade in the credit rating of the firm even if it does not lose market value and, as Fig. 6 indicates, even with a strictly positive growth (point C). In order to keep its credit rating a firm value growth large enough (point B), has to take place. In summary, any ratio $V(\tau)/\psi$ lower that B will be followed by a downgrade, while any value of this ratio above B will lead to an upgrade. Note also that deviations from B will have a different impact on the credit rating depending on whether this deviation is positive or negative. A negative deviation will have a higher effect in absolute terms

than an equivalent positive deviation. Debt refinancing therefore is expected to generate stronger downgrades than upgrades.

3.2. $n=2$

We have analyzed the case in which the firm always maintains a single zero coupon bond as corporate debt. The main implications derived from assuming that the firm refinances its debt in terms of credit ratings and credit spreads appear in this simple case. Exploring the situation in which the firm always refinances with $n=2$ is interesting however for several reasons. First, it can be seen as a simplification to short and long term debt, what better represents the debt structure of a firm. Second, it incorporates the fact that equity holders do not only care about the debt that currently matures at the time of deciding whether or not satisfying it, but also about all future debt remaining. This makes for instance the current bankruptcy-triggering firm value to diverge from the current nominal debt payment, something that does not happen with $n=1$. Proposition 1 ensures that at any time the firm has to satisfy a debt payment, refinancing with $n=2$ is feasible.

Proposition 1. B1–B3 hold for $n=2$, making a refinancing contract with $n=2$ feasible whenever $S(V,\tau) > 0$.

Proof. See Appendix A.

Proposition 1 does not assume any specific initial debt structure. However, we may think in a model in which the firm maintains a stable corporate debt structure with short and long term debt, keeping the ratio short term debt/long term debt, and the time spread between them, constant. This stable corporate debt structure translates into a vector $\Delta \equiv (n, \Phi, \Lambda)$, where $n=2$, $\Phi = (1, \phi)$ and $\Lambda = (1, \eta)$. Assume again that the firm does not pay dividends. As a result, it is always possible to consider that the firm refinances not only under a constant Δ (as stated in the theorem), but also with some fixed T_1 (see Corollary 2, S1). The following proposition implies that the effect of debt refinancing on the evolution of the credit rating of the firm described for $n=1$, also applies in this case.

Proposition 2. Let \bar{V} and \bar{V}_1 denote the bankruptcy-triggering firm value at τ and τ_1 respectively. Then, \bar{V}_1 is a strictly decreasing and strictly convex function in $V(\tau)$, with $\lim_{V(\tau) \rightarrow \bar{V}} \bar{V}_1 = \infty$ and $\lim_{V(\tau) \rightarrow \infty} \bar{V}_1 = \bar{V}e^{rT_1}$.

Proof. See Appendix A.

¹⁰ Merton finds that the sign of $C.S_{\tau_1}$ depends on whether q is higher, equal or lower than 1.

¹¹ To simplify the exposition we assume a non dividend paying firm. The strict convexity follows from $(\psi_1)_V < 0$ and $(\psi_1)_{VV} > 0$.

The shape of \bar{V}_1 as a function of $V(\tau)$ will be therefore analogous to the one we found for ψ_1 with respect to $V(\tau)$ under $n=1$. The credit rating at τ will be now described by the ratio $V(\tau)/\bar{V}_1$, and the same analysis we made for $V(\tau)/\psi_1$ applies in this case. Refinancing makes \bar{V}_1 to be the new bankruptcy-triggering firm value, a critical threshold that will evolve over time as the firm refinances its debt repeatedly. Although no explicit solution for it can be provided, it can be shown to belong to the same range in which KMV typically finds to be the default point.

Proposition 3. *Let ψ_1 and ψ_2 be the new short and long term debt resulting from refinancing at τ , and let T be the time spread between these payments. Then, for any $\sigma \in (0, \infty)$, $\bar{V}_1 \in (\psi_1, \psi_1 + \psi_2 e^{-rT})$. Moreover, $\lim_{\sigma \rightarrow 0} \bar{V}_1 = \psi_1 + \psi_2 e^{-rT}$ and $\lim_{\sigma \rightarrow \infty} \bar{V}_1 = \psi_1$.*

Proof. See Appendix A.

With a database of over 100,000 company-years and over 2000 incidents of default or bankruptcy, KMV has found that firms generally default when the firm value lies somewhere between short term debt and total debt in nominal terms.¹² Clearly, $\bar{V}_1 \in (\psi_1, \psi_1 + \psi_2 e^{-rT})$ implies that $\bar{V}_1 \in (\psi_1, \psi_1 + \psi_2)$.

4. Conclusions

Refinancing existing debt seems to be one of the most important, if not the first, reason to issue new debt. We investigate the long run effects of this extended practice on credit ratings and credit spreads. Debt refinancing generates systematic rating downgrades unless a minimum firm value growth is observed. Deviations from this growth path imply asymmetric results. A lower firm value growth generates downgrades and a higher firm value growth upgrades as expected. However, downgrades will tend to be higher in absolute terms. Finally, the risk free rate will have a negative effect over the credit spreads in the short run, a null effect in the medium run, and a positive effect in the long run.

Acknowledgements

Forte acknowledges financial support from MEC Grant Ref: AP2000-1327 and Banco Sabadell. Peña acknowledges financial support from MEC grants PB98-0030 and BEC2002-00279, and MCI grant ECO2009-12551. Preliminary versions of this paper have been presented with the title “Debt Financing and Default Probabilities” at the IX Foro de Finanzas (Pamplona), and with the title “The Design of Refinancing Contracts” at LACEA 2002. The authors appreciate helpful comments from Carmen Ansotegui and an anonymous referee. The usual disclaimers apply.

Appendix A.

A.1. Proof of lemma

$S(V, \Theta, \tau)$ is what equity holders get after signing Θ , therefore, they will be willing to sign if and only if $S(V, \Theta, \tau) \geq S(V, \tau)$. $F(V, \Theta, \tau)$ is what debt holders have after signing Θ , therefore, they will be willing to sign if and only if $F(V, \Theta, \tau) \geq F(V, \tau)$. At the same time $S(V, \Theta, \tau) + F(V, \Theta, \tau) = S(V, \tau) + F(V, \tau) = V$. $S(V, \Theta, \tau) = S(V, \tau)$ implies that $F(V, \Theta, \tau) = F(V, \tau)$ and $\Theta \in \Theta^F$. On the other hand $\Theta \in \Theta^F$ implies $S(V, \Theta, \tau) \geq S(V, \tau)$. Suppose $S(V, \Theta, \tau) > S(V, \tau)$, then $F(V, \Theta, \tau) < F(V, \tau)$ and $\Theta \notin \Theta^F$, what is a contradiction, proving the first argument of

the lemma. Finally, $S(V, \Theta, \tau) > 0 \forall \Theta | \tau_1 > \tau$, implying $S(V, \tau) > 0$ as a necessary condition for a feasible Θ to exist.¹³

A.2. Proof of theorem

We have six possible sequences α – β – γ . Let us analyze each one of them:

Case 1 ($\tau_1 - \delta - \psi_1$). Given $\tau_1 \in \varphi^{\tau_1}$, $S(V, \Theta, \tau)$ is CSD (ψ_1), with $\lim_{\psi_1 \rightarrow \hat{\psi}_1} S(V, \Theta, \tau) > S(V, \tau) \forall \delta \in \varphi^\delta$, and $\lim_{\psi_1 \rightarrow \infty} S(V, \Theta, \tau) = V(1 - e^{-\delta \tau_1})$. Therefore, $S(V, \Theta, \tau) = S(V, \tau)$ reaches a solution for at least one $\psi_1 \in \varphi^{\psi_1}$, if and only if $V(1 - e^{-\delta \tau_1}) < S(V, \tau)$, if and only if $\delta < \ln[V/F(V, \tau)]/\tau_1$. As a result, $\varphi^{\delta | \tau_1} \equiv [0, \ln[V/F(V, \tau)]/\tau_1) \neq \emptyset$. Because $S(V, \Theta, \tau)$ is CSD (ψ_1), we finally have that for any $\tau_1 \in \varphi^{\tau_1}$, and $\delta \in \varphi^{\delta | \tau_1}$, there is a unique $\psi_1 \in \varphi^{\psi_1}$ such that $S(V, \Theta, \tau) = S(V, \tau)$.

Case 2 ($\tau_1 - \psi_1 - \delta$). Given $\tau_1 \in \varphi^{\tau_1}$, $S(V, \Theta, \tau)$ is CSI (δ), with $\lim_{\delta \rightarrow \infty} S(V, \Theta, \tau) = V \forall \psi_1 \in \varphi^{\psi_1}$. Therefore, $S(V, \Theta, \tau) = S(V, \tau)$ reaches a solution for at least one $\delta \in \varphi^\delta$, if and only if $S(V, \Theta, \tau)|_{\delta=0} \leq S(V, \tau)$, if and only if $\psi_1 \geq \psi_1^{0, \tau_1}$, where $\psi_1^{0, \tau_1} > \hat{\psi}_1$ is the ψ_1 value such that $S(V, \Theta, \tau)|_{\delta=0} = S(V, \tau)$.¹⁴ As a result, $\varphi^{\psi_1 | \tau_1} \equiv [\psi_1^{0, \tau_1}, \infty) \neq \emptyset$. Because $S(V, \Theta, \tau)$ is CSI (δ), we finally have that for any $\tau_1 \in \varphi^{\tau_1}$, and $\psi_1 \in \varphi^{\psi_1 | \tau_1}$, there is a unique $\delta \in \varphi^\delta$ such that $S(V, \Theta, \tau) = S(V, \tau)$.

Case 3 ($\delta - \tau_1 - \psi_1$). Given $\delta \in \varphi^\delta$, $S(V, \Theta, \tau)$ is CSD (ψ_1), with $\lim_{\psi_1 \rightarrow \hat{\psi}_1} S(V, \Theta, \tau) > S(V, \tau) \forall \tau_1 \in \varphi^{\tau_1}$, and $\lim_{\psi_1 \rightarrow \infty} S(V, \Theta, \tau) = V(1 - e^{-\delta \tau_1})$. Therefore, $S(V, \Theta, \tau) = S(V, \tau)$ reaches a solution for at least one $\psi_1 \in \varphi^{\psi_1}$, if and only if $V(1 - e^{-\delta \tau_1}) < S(V, \tau)$, if and only if $\tau_1 < \tau + \ln[V/F(V, \tau)]/\delta$. As a result, $\varphi^{\tau_1 | \delta} \equiv (\tau, \tau + \ln[V/F(V, \tau)]/\delta) \neq \emptyset$. Because $S(V, \Theta, \tau)$ is CSD (ψ_1), we finally have that for any $\delta \in \varphi^\delta$, and $\tau_1 \in \varphi^{\tau_1 | \delta}$, there is a unique $\psi_1 \in \varphi^{\psi_1}$ such that $S(V, \Theta, \tau) = S(V, \tau)$.

Case 4 ($\delta - \psi_1 - \tau_1$). Given $\delta \in \varphi^\delta$, $S(V, \Theta, \tau)$ is CSI (τ_1), with $\lim_{\tau_1 \rightarrow \tau} S(V, \Theta, \tau) = V - \psi_1 \sum_{i=1}^n \phi_i < S(V, \tau) \forall \psi_1 \in \varphi^{\psi_1}$, and $\lim_{\tau_1 \rightarrow \infty} S(V, \Theta, \tau) = V$. As a result, $\varphi^{\psi_1 | \delta} \equiv \varphi^{\psi_1} \neq \emptyset$. Because $S(V, \Theta, \tau)$ is CSI (τ_1), we finally have that for any $(\delta, \psi_1) \in \varphi^\delta \times \varphi^{\psi_1}$, there is a unique $\tau_1 \in \varphi^{\tau_1}$ such that $S(V, \Theta, \tau) = S(V, \tau)$.

Case 5 ($\psi_1 - \tau_1 - \delta$). Given $\psi_1 \in \varphi^{\psi_1}$, $S(V, \Theta, \tau)$ is CSI (δ), with $\lim_{\delta \rightarrow \infty} S(V, \Theta, \tau) = V \forall \tau_1 \in \varphi^{\tau_1}$. Therefore, $S(V, \Theta, \tau) = S(V, \tau)$ reaches a solution for at least one $\delta \in \varphi^\delta$, if and only if $S(V, \Theta, \tau)|_{\delta=0} \leq S(V, \tau)$, if and only if $\tau_1 \leq \tau_1^{0, \psi_1}$, where $\tau_1^{0, \psi_1} > \tau$ is the τ_1 value such that $S(V, \Theta, \tau)|_{\delta=0} = S(V, \tau)$.¹⁵ As a result, $\varphi^{\tau_1 | \psi_1} \equiv (\tau, \tau_1^{0, \psi_1}) \neq \emptyset$. Because $S(V, \Theta, \tau)$ is CSI (δ), we finally have that for any $\psi_1 \in \varphi^{\psi_1}$, and $\tau_1 \in \varphi^{\tau_1 | \psi_1}$, there is a unique $\delta \in \varphi^\delta$ such that $S(V, \Theta, \tau) = S(V, \tau)$.

Case 6 ($\psi_1 - \delta - \tau_1$). Given $\psi_1 \in \varphi^{\psi_1}$, $S(V, \Theta, \tau)$ is CSI (τ_1), with $\lim_{\tau_1 \rightarrow \tau} S(V, \Theta, \tau) = V - \psi_1 \sum_{i=1}^n \phi_i < S(V, \tau) \forall \delta \in \varphi^\delta$, and $\lim_{\tau_1 \rightarrow \infty} S(V, \Theta, \tau) = V$. As a result, $\varphi^{\delta | \psi_1} \equiv \varphi^\delta \neq \emptyset$. Because $S(V, \Theta, \tau)$ is CSI (τ_1), we finally have that for any $(\psi_1, \delta) \in \varphi^{\psi_1} \times \varphi^\delta$, there is a unique $\tau_1 \in \varphi^{\tau_1}$ such that $S(V, \Theta, \tau) = S(V, \tau)$.

A.3. Proof of Corollary 2

This holds for S1, S2 and S3, given the proof of the theorem for cases 1 and 3, 2 and 5, and 4 and 6, respectively.

A.4. Proof of Proposition 1

Let $T_1 = \tau_1 - \tau$ and $T_2 = \tau_2 - \tau$. Following the restriction imposed in Section 2 we will express ψ_2 as $\phi\psi_1$, and T_2 as $(1 + \eta)T_1$. The

¹³ Limited liability makes $S(V, \Theta, \tau) > 0$ when no payment has to be currently satisfied.

¹⁴ B1 implies that ψ_1^{0, τ_1} exists and is unique, and joint with B2 also implies that $\psi_1^{0, \tau_1} > \hat{\psi}_1$.

¹⁵ B2 implies that τ_1^{0, ψ_1} exists and is unique. B2 also implies that $\tau_1^{0, \psi_1} > \tau$.

¹² Crosbie (1999).

equity value is given in this case by

$$S(V, \Theta, \tau) = V(1 - e^{-\delta T_1}) + Ve^{-\delta T_1}(1 - e^{-\delta \eta T_1})N(a_1) \\ + Ve^{-\delta(1+\eta)T_1}N_2(a_1, b_1; \rho) \\ - \phi \psi_1 e^{-r(1+\eta)T_1}N_2(a_2, b_2; \rho) - \psi_1 e^{-rT_1}N(a_2),$$

where $N_2(a, b; \rho)$ represents the standard bivariate normal distribution function, with integration limits a and b , and correlation coefficient ρ . In addition,

$$a_1 = \frac{\ln(V/\bar{V}_1) + (r - \delta + (\sigma^2/2))T_1}{\sigma \sqrt{T_1}},$$

$$a_2 = a_1 - \sigma \sqrt{T_1},$$

$$b_1 = \frac{\ln(V/\phi \psi_1) + (r - \delta + (\sigma^2/2))(1 + \eta)T_1}{\sigma \sqrt{(1 + \eta)T_1}},$$

$$b_2 = b_1 - \sigma \sqrt{(1 + \eta)T_1},$$

$$\rho = \sqrt{\frac{1}{1 + \eta}},$$

where \bar{V}_1 is the firm value that satisfies

$$S(\bar{V}_1, \Theta, \tau_1) = \bar{V}_1(1 - e^{-\delta \eta T_1}) + \bar{V}_1 e^{-\delta \eta T_1} N(c_1) \\ - \phi \psi_1 e^{-r \eta T_1} N(c_2) - \psi_1 = 0,$$

with

$$c_1 = \frac{\ln(\bar{V}_1/\phi \psi_1) + (r - \delta + (\sigma^2/2))\eta T_1}{\sigma \sqrt{\eta T_1}},$$

$$c_2 = c_1 - \sigma \sqrt{\eta T_1}.$$

$S(V, \Theta, \tau)$ is then a continuous function in ψ_1 , with

$$S(V, \Theta, \tau)_{\psi_1} = -[\phi e^{-r(1+\eta)T_1}N_2(a_2, b_2; \rho) + e^{-rT_1}N(a_2)] < 0.$$

This, joint with

$$S(V, \Theta, \tau)|_{\psi_1=0} = V,$$

$$\lim_{\psi_1 \rightarrow \infty} S(V, \Theta, \tau) = V(1 - e^{-\delta T_1}),$$

proves that Assumption B1 holds.

At the same time $S(V, \Theta, \tau)$ is a continuous function in τ_1 , with

$$S(V, \Theta, \tau)_{\tau_1} = \delta V e^{-\delta T_1} [1 - N(a_1)] + \delta(1 + \eta) V e^{-\delta(1+\eta)T_1} [N(a_1) \\ - N_2(a_1, b_1; \rho)] + r(1 + \eta) \phi \psi_1 e^{-r(1+\eta)T_1} N_2(a_2, b_2; \rho) \\ + \phi \psi_1 e^{-r(1+\eta)T_1} f(a_2) N(c_2) \frac{\sigma}{2\sqrt{T_1}} \\ + \phi \psi_1 e^{-r(1+\eta)T_1} f(b_2) N\left(\frac{a_2 - \rho b_2}{\sqrt{1 - \rho^2}}\right) \frac{\sigma}{2\sqrt{(1 + \eta)T_1}} \\ + r \psi_1 e^{-rT_1} N(a_2) + \psi_1 e^{-rT_1} f(a_2) \frac{\sigma}{2\sqrt{T_1}} > 0$$

$$\lim_{\tau_1 \rightarrow \infty} S(V, \Theta, \tau) = V$$

$$\lim_{\tau_1 \rightarrow \tau} S(V, \Theta, \tau) = \begin{cases} V - \psi_1 - \phi \psi_1 & \text{if } V > \psi_1 + \phi \psi_1 \\ 0 & \text{if } V \leq \psi_1 + \phi \psi_1 \end{cases} \\ = \text{Max}\{0, V - \psi_1 - \phi \psi_1\}$$

proving that Assumption B2 also holds.

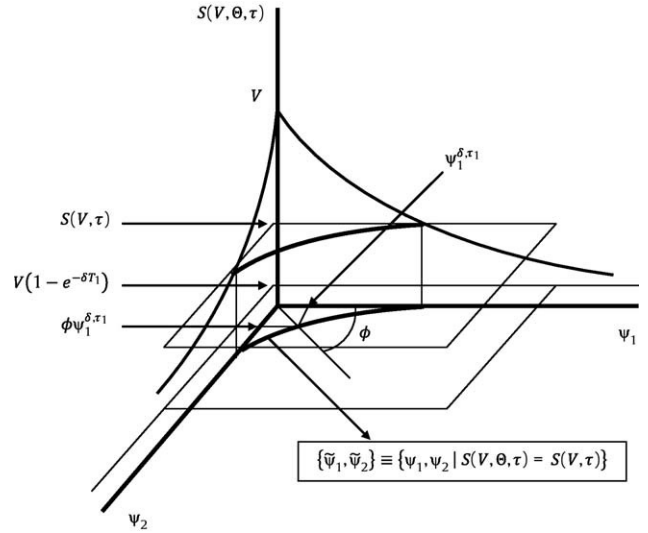


Fig. 7. This figure shows the sequence $\tau_1 - \delta - \psi_1$ for $n=2$.

Finally, $S(V, \Theta, \tau)$ is a continuous function in δ , with

$$S(V, \Theta, \tau)_{\delta} = T_1 V e^{-\delta T_1} [1 - N(a_1)] + (1 + \eta) T_1 V e^{-\delta(1+\eta)T_1} \\ \times [N(a_1) - N_2(a_1, b_1; \rho)] > 0$$

$$\lim_{\delta \rightarrow \infty} S(V, \Theta, \tau) = V$$

implying that Assumption B3 is equally satisfied.¹⁶

It is possible to analyze graphically case 1 for $n=2$. Let denote $S(V, \Theta, \tau)$ simply as S , and let show that S is strictly convex in (ψ_1, ψ_2) , where we do not impose the restriction $\psi_2 = \phi \psi_1$.

$$S_{\psi_1} = -e^{-rT_1} N(a_2),$$

$$S_{\psi_2} = -e^{-rT_2} N_2(a_2, b_2; \rho),$$

$$S_{\psi_1 \psi_1} = \frac{e^{-rT_1} f(a_2)}{\bar{V}_1 \sigma \sqrt{T_1} \{1 - e^{-\delta T} [1 - N(c_1)]\}} > 0,$$

$$S_{\psi_2 \psi_2} = \frac{e^{-r(T_2+T)} f(a_2) [N(c_2)]^2}{\bar{V}_1 \sigma \sqrt{T_1} \{1 - e^{-\delta T} [1 - N(c_1)]\}} \\ + \frac{e^{-rT_2} f(b_2) N[(a_2 - \rho b_2)/\sqrt{1 - \rho^2}]}{\psi_2 \sigma \sqrt{T_2}} > 0,$$

$$S_{\psi_1 \psi_2} = \frac{e^{-rT_2} f(a_2) N(c_2)}{\bar{V}_1 \sigma \sqrt{T_1} \{1 - e^{-\delta T} [1 - N(c_1)]\}},$$

$$S_{\psi_1 \psi_1} S_{\psi_2 \psi_2} - (S_{\psi_1 \psi_2})^2 = \frac{e^{-r(T_1+T_2)} f(a_2) f(b_2) N[(a_2 - \rho b_2)/\sqrt{1 - \rho^2}]}{\bar{V}_1 \psi_2 \sigma^2 \sqrt{T_1} \sqrt{T_2} \{1 - e^{-\delta T} [1 - N(c_1)]\}} \\ > 0.$$

The strict convexity of S , joint with $S|_{(\psi_1, \psi_2)=(0,0)} = V$, and $\lim_{(\psi_1, \psi_2) \rightarrow (\infty, \infty)} S = V(1 - e^{-\delta T_1})$, leads to Fig. 7.

¹⁶ A more detailed description of the steps followed in this proof is available upon request.

A.5. Proof of Proposition 2

Let ψ be the debt payment to be satisfied at τ . Then

$$S(V, \tau) = VN(w_1) - \phi\psi e^{-r\eta T_1} N(w_2) - \psi = 0,$$

where

$$w_1 = \frac{\ln(V/\phi\psi) + (r + (\sigma^2/2))\eta T_1}{\sigma\sqrt{\eta T_1}},$$

$$w_2 = w_1 - \sigma\sqrt{\eta T_1},$$

and $S(V, \tau) > 0 \forall V > \bar{V}$, being \bar{V} the implicit solution to $S(\bar{V}, \tau) = 0$.

On the other hand,

$$S(V, \Theta, \tau) = VN_2(k_1, l_1; \rho) - \phi\psi_1 e^{-r(1+\eta)T_1} N_2(k_2, l_2; \rho) - \psi_1 e^{-rT_1} N(k_2),$$

where

$$k_1 = \frac{\ln(V/\bar{V}_1) + (r + (\sigma^2/2))T_1}{\sigma\sqrt{T_1}},$$

$$k_2 = k_1 - \sigma\sqrt{T_1},$$

$$l_1 = \frac{\ln(V/\phi\psi_1) + (r + (\sigma^2/2))(1 + \eta)T_1}{\sigma\sqrt{(1 + \eta)T_1}},$$

$$l_2 = l_1 - \sigma\sqrt{(1 + \eta)T_1},$$

$$\rho = \sqrt{\frac{1}{1 + \eta}},$$

and \bar{V}_1 is the implicit solution to

$$S(\bar{V}_1, \Theta, \tau_1) = \bar{V}_1 N(h_1) - \phi\psi_1 e^{-r\eta T_1} N(h_2) - \psi_1 = 0,$$

with

$$h_1 = \frac{\ln(\bar{V}_1/\phi\psi_1) + (r + (\sigma^2/2))\eta T_1}{\sigma\sqrt{\eta T_1}},$$

$$h_2 = h_1 - \sigma\sqrt{\eta T_1}.$$

If we define $\theta = \bar{V}_1/\psi_1$, then previous expression can be written as

$$\theta N(h_1) - \phi e^{-r\eta T_1} N(h_2) - 1 = 0,$$

where

$$h_1 = \frac{\ln(\theta/\phi) + (r + (\sigma^2/2))\eta T_1}{\sigma\sqrt{\eta T_1}}.$$

The result is that θ is a constant, that is, θ will not depend on the firm value at τ (although \bar{V}_1 and ψ_1 will). Note also that $\theta = \bar{V}/\psi$. We can express condition $\Gamma(V, \Theta, \tau) = S(V, \Theta, \tau) - S(V, \tau) = 0$ as

$$\begin{aligned} \Gamma(V, \Theta, \tau) &= [VN_2(k_1, l_1; \rho) - \bar{V}_1(\phi/\theta)e^{-r(1+\eta)T_1} N_2(k_2, l_2; \rho) \\ &\quad - \bar{V}_1(1/\theta)e^{-rT_1} N(k_2)] - [VN(w_1) - \bar{V}(\phi/\theta)e^{-r\eta T_1} N(w_2) \\ &\quad - \bar{V}(1/\theta)] = 0, \end{aligned}$$

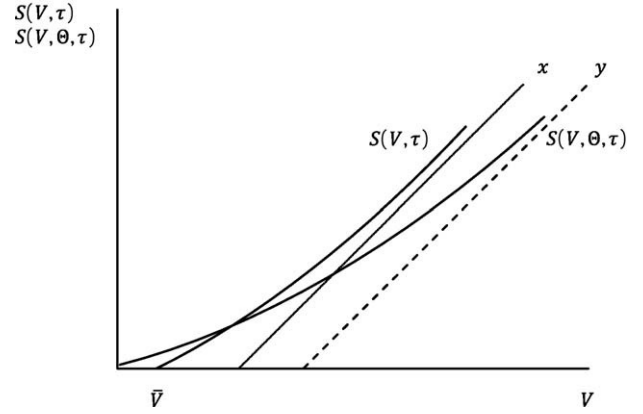


Fig. 8. This figure shows $S(V, \tau)$ and $S(V, \Theta, \tau)$ as a function of V for $n=2$.

where

$$w_1 = \frac{\ln(V/\bar{V}) + \ln(\theta/\phi) + (r + (\sigma^2/2))\eta T_1}{\sigma\sqrt{\eta T_1}},$$

$$l_1 = \frac{\ln(V/\bar{V}_1) + \ln(\theta/\phi) + (r + (\sigma^2/2))(1 + \eta)T_1}{\sigma\sqrt{(1 + \eta)T_1}}.$$

This condition implies that as V tends to \bar{V} , \bar{V}_1 tends to infinity. At the same time, $\lim_{V \rightarrow \infty} \bar{V}_1 = \bar{V}e^{rT_1}$.¹⁷ It can also be proved that \bar{V}_1 is a strictly decreasing function in V . In fact,

$$\begin{aligned} (\bar{V}_1)_V &= -\frac{\Gamma(V, \Theta, \tau)_V}{\Gamma(V, \Theta, \tau)_{\bar{V}_1}} \\ &= -\frac{N(w_1) - N_2(k_1, l_1; \rho)}{(\phi/\theta)e^{-r(1+\eta)T_1} N_2(k_2, l_2; \rho) + (1/\theta)e^{-rT_1} N(k_2)} < 0. \end{aligned}$$

Fig. 8 represents $S(V, \tau)$ and $S(V, \Theta, \tau)$ as a function of V .¹⁸ Clearly, for any $V > \bar{V}$, $N(w_1) > N_2(k_1, l_1; \rho)$, given that these are the derivatives of $S(V, \tau)$ and $S(V, \Theta, \tau)$ with respect to V . This implies $(\bar{V}_1)_V < 0$. $(\bar{V}_1)_{VV} > 0$ finally follows from the fact that $(\phi/\theta)e^{-r(1+\eta)T_1} N_2(k_2, l_2; \rho) + (1/\theta)e^{-rT_1} N(k_2)$ is a strictly increasing function in V , while $N(w_1) - N_2(k_1, l_1; \rho)$ is strictly decreasing $\forall V > \bar{V}$ as Fig. 8 indicates, and actually tends to zero as V grows.

A.6. Proof of Proposition 3

Let

$$Z(V) = VN(g_1) - \psi_2 e^{-rT} N(g_2) - \psi_1,$$

where

$$g_1 = \frac{\ln(V/\psi_2) + (r + (\sigma^2/2))T}{\sigma\sqrt{T}},$$

$$g_2 = g_1 - \sigma\sqrt{T},$$

then, $\bar{V}_1 \equiv V|Z(V) = 0$.

¹⁷ Note that this implies that $\lim_{V \rightarrow \infty} \psi_1 = \psi e^{rT_1}$ and $\lim_{V \rightarrow \infty} \phi\psi_1 = \phi\psi e^{rT_1}$, that is, as the default risk tends to zero, new debt payments tend to current payments capitalized at the risk free interest rate.

¹⁸ $x = V - \bar{V}(\phi/\theta)e^{-r\eta T_1} - \bar{V}(1/\theta)$ and $y = V - \bar{V}_1(\phi/\theta)e^{-r(1+\eta)T_1} - \bar{V}_1(1/\theta)e^{-rT_1}$. $\bar{V}(\phi/\theta)e^{-r\eta T_1} + \bar{V}(1/\theta) < \bar{V}_1(\phi/\theta)e^{-r(1+\eta)T_1} + \bar{V}_1(1/\theta)e^{-rT_1}$ given $\bar{V}_1 > \bar{V}e^{rT_1}$.

Suppose first $\sigma \in (0, \infty)$ and $V = \psi_1$; then

$$\begin{aligned} Z(\psi_1) &= \psi_1 N(g_1) - \psi_2 e^{-rT} N(g_2) - \psi_1 \\ &= -\psi_1 [1 - N(g_1)] - \psi_2 e^{-rT} N(g_2) < 0. \end{aligned}$$

This, joint with $Z_V = N(g_1) > 0$, implies that $\bar{V}_1 > \psi_1 \quad \forall \sigma \in (0, \infty)$. We have that $\lim_{\sigma \rightarrow \infty} N(g_1) = 1$ and $\lim_{\sigma \rightarrow \infty} N(g_2) = 0$, therefore, $\lim_{\sigma \rightarrow \infty} Z(V) = V - \psi_1 = 0 \Leftrightarrow V = \psi_1$, proving $\lim_{\sigma \rightarrow \infty} \bar{V}_1 = \psi_1$.

On the other hand, consider $V = \psi_1 + \psi_2 e^{-rT}$; then

$$Z(\psi_1 + \psi_2 e^{-rT}) = -\psi_1 [1 - N(g_1)] + \psi_2 e^{-rT} [N(g_1) - N(g_2)].$$

$\lim_{\sigma \rightarrow 0} Z(\psi_1 + \psi_2 e^{-rT}) = 0$ because $\lim_{\sigma \rightarrow 0} N(g_1 | V = \psi_1 + \psi_2 e^{-rT}) = \lim_{\sigma \rightarrow 0} N(g_2 | V = \psi_1 + \psi_2 e^{-rT}) = 1$. As a result, $\bar{V}_1 = \psi_1 + \psi_2 e^{-rT}$ in this limit case. Given that $Z_\sigma > 0$ we also have that $Z(\psi_1 + \psi_2 e^{-rT}) > 0 \quad \forall \sigma \in (0, \infty)$. This, joint again with $Z_V = N(g_1) > 0$, implies $\bar{V}_1 < \psi_1 + \psi_2 e^{-rT} \quad \forall \sigma \in (0, \infty)$, and concludes the proof.

References

- Altman, E.I., Kao, D.L., 1992. Rating drift in high yield bonds. *Journal of Fixed Income* 1, 15–20.
- Crosbie, P.J., 1999. Modeling Default Risk. KMV Corporation.
- Gande, A., Puri, M., Saunders, A., Walter, I., 1997. Bank underwriting of debt securities: modern evidence. *Review of Financial Studies* 10, 1175–1202.
- Geske, R., 1977. The valuation of corporate liabilities as compound options. *Journal of Financial and Quantitative Analysis* 12, 541–552.
- Guha, D., Hiris, L., Visviki, D., 2001. Forecasting the quality spread using business cycle indicators, Unpublished document.
- Hansen, R.S., Crutchley, C., 1990. Corporate earnings and financings: an empirical analysis. *Journal of Business* 63, 347–371.
- Leland, H.E., 1994. Corporate debt value, bond covenants, and optimal capital structure. *Journal of Finance* 49, 1213–1252.
- Longstaff, F.A., Schwartz, E.S., 1995. A simple approach to valuing risky fixed and floating rate debt. *Journal of Finance* 50, 789–819.
- Merton, R.C., 1974. On the pricing of corporate debt: the risk structure of interest rates. *Journal of Finance* 29, 449–470.